Published by Institute of Physics Publishing for SISSA/ISAS

Received: *December 7, 2004*
Revised: *January 24, 2005*
Accepted: *February 1, 2005*

Topological open string amplitudes on orientifolds

Vincent Bouchard

Mathematical Institute, University of Oxford
24-29 St. Giles', Oxford OX1 3LB, England, U.K.
E-mail: bouchard@maths.ox.ac.uk

Bogdan Florea

Department of Physics and Astronomy, Rutgers University
Piscataway, NJ 08855-0849, U.S.A.
E-mail: florea@physics.rutgers.edu

Marcos Mariño*

Department of Physics, CERN
Geneva 23, CH-1211, Switzerland
E-mail: marcos@mail.cern.ch

Abstract: We study topological open string amplitudes on orientifolds without fixed planes. We determine the contributions of the untwisted and twisted sectors as well as the BPS structure of the amplitudes. We illustrate our general results in various examples involving D-branes in toric orientifolds. We perform the computations by using both the topological vertex and unoriented localization. We also present an application of our results to the BPS structure of the coloured Kauffman polynomial of knots.

Keywords: Chern-Simons Theories, Topological Strings.

*Also at Departamento de Matemática, IST, Lisboa, Portugal.

JHEP02(2005)002

Contents

1. Introduction

Geometric or large N transitions relating open and closed topological string theories [13] have had a deep impact in the study of topological strings, and they have led to the first systematic solution of these models on noncompact, toric Calabi-Yau threefolds through the topological vertex of [2] (see [31, 36] for a review). The study of topological strings on Calabi-Yau orientifolds was initiated in [40], where an orientifold of the geometric transition of [13] relating the deformed and the resolved conifold was studied in detail, and continued in [1] from the B-model point of view. The geometric transition of [13] can be extended to more general toric geometries [4, 7, 8, 3], and in [5] we proposed in fact a general class of large N dualities involving orientifolds of non-compact toric Calabi-Yau threefolds. These dualities involve $\mathrm{U}(N)$, $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ Chern-Simons gauge theories, and they make possible the computation of unoriented string amplitudes. The results obtained through large N dualities were also checked in [5] against independent localization computations. Moreover, we found a topological vertex prescription to compute these amplitudes directly, extending in this way the general formalism of the topological vertex to include the case of orientifolds without fixed planes.

In this paper we continue the study of topological string amplitudes on orientifolds initiated in [5]. Our main goal is to extend the results in [5] to topological open strings on orientifolds without fixed points. In other words, we consider orientifolds of non-compact Calabi-Yau threefolds with D-branes.

An important property of topological string amplitudes is that they have an integrality structure related to the counting of BPS states, as it was first realized by Gopakumar and Vafa [14] in the case of closed string amplitudes. The integrality structure in the open case was studied in [38, 25]. As a first step in our study of topological string amplitudes on orientifolds without fixed planes we analyze their BPS structure. What we find is that the total orientifold amplitude is the sum of an oriented amplitude (the untwisted sector) and an unoriented amplitude (the twisted sector) with different integrality properties. We explain how to compute the contribution of the twisted sector in the open case. We also spell out in detail the integrality properties of the twisted sector contributions.

This integrality structure provides a strong requirement on topological open string amplitudes, and we check it explicitly on various examples involving orientifolds with D-branes. To compute these open string amplitudes we use the new vertex rule introduced in [5]. We also compute the associated Gromov-Witten invariants using independent localization techniques developed in [9, 5], and find perfect agreement with the results obtained with the vertex.

One of the most interesting applications of the large N duality between open and closed topological strings consists in the determination of structural properties of knot and link invariants related to the BPS structure of open topological strings. For example, from the results of [38, 25] one can deduce structure theorems for the coloured HOMFLY polynomial of knots and links. The large N duality on orientifolds now involves $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ Chern-Simons theories. Therefore, the BPS structure of the amplitudes should lead to the determination of structural properties of a different type of knot and link invariant: the coloured Kauffman polynomial [20]. Although for arbitrary knots and links we cannot determine in detail the structure of the untwisted sector, we are able to derive general structural results for the coloured Kauffman polynomial. We test again these predictions on various examples involving torus knots.

The paper is structured as follows. In section 2, we explore the BPS content of closed and open topological string amplitudes on orientifolds and formulate their structural properties. We then compute explicitly the amplitudes for various examples in section 3: the SO / Sp framed unknot, the SO / Sp framed Hopf link, and an outer brane in $\mathbb{P}^2$ attached to $\mathbb{RP}^2$. The independent localization computations we provide for all these examples corroborate our methods and proposals. In section 4 we formulate structural properties of the coloured Kauffman polynomial. We discuss our results and propose new avenues of research in section 5. Finally, appendix A contains useful formulae, while in appendix B we give a full proof of the identity that was conjectured (and partially proved) in [5]; this

JHEP02(2005)002

identity shows that the new vertex rule introduced in [5] to compute amplitudes on orientifolds agrees with the results of large N SO / Sp transitions. Appendix C lists some results for BPS invariants coming from SO Chern-Simons theory.

2. Topological open string amplitudes in orientifolds

2.1 BPS structure of topological string amplitudes

One of the most important results of topological string theory is the fact that topological string amplitudes have an integrality, or BPS structure, which expresses them in terms of numbers of BPS states. Let us briefly review the known results for both open and closed strings.

In the case of topological closed strings on Calabi-Yau threefolds, the BPS structure was obtained by Gopakumar and Vafa in [14]. Let us denote by $F_g(t)$ the topological string free energy at genus g, where t denotes the set of Kähler parameters of the Calabi-Yau threefold X, and let

$$\mathcal{F}(t, g_s) = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t) \tag{2.1}$$

be the total free energy. Then, one has the following structure result:

$$\mathcal{F}(t, g_s) = \sum_{d=1}^{\infty} \sum_{g=0}^{\infty} \sum_{\beta} \frac{1}{d} \frac{n_\beta^g}{(q^{d/2} - q^{-d/2})^{2-2g}} e^{-d\beta\cdot t}\,, \tag{2.2}$$

where $q = e^{ig_s}$, the sum over β is over two-homology classes in X, and n_β^g (the so-called Gopakumar-Vafa invariants) are *integers*. The factor $(q^{d/2}-q^{-d/2})^{2g}$ comes from computing a signed trace over the space of differential forms on a Riemann surface of genus g, while the factor $(q^{d/2} - q^{-d/2})^{-2}$ comes from a Schwinger computation [14].

For open string amplitudes, the structure of the amplitudes was found in [38, 25] and is much more delicate. To define an open string amplitude we have to specify boundary conditions through a set of submanifolds of X, $S_1, \dots, S_L$. To each of these submanifolds we associate a source V_ℓ, $\ell = 1, \dots, L$, which is a $\mathrm{U}(M)$ matrix. The total partition function is given by

$$Z(V_1, \dots, V_L) = \sum_{R_1,\cdots,R_L} Z_{(R_1,\cdots,R_L)} \prod_{\alpha=1}^{L} \mathrm{Tr}_{R_\alpha} V_\alpha\,, \tag{2.3}$$

where R_α denote representations of $\mathrm{U}(M)$ and we are considering the limit $M \to \infty$. The amplitudes $Z_{(R_1,\dots,R_L)}$ can be computed in the noncompact, toric case by using the topological vertex [2]. According to the correspondence proposed in [38], they are given in some cases by invariants of links whose components are coloured by representations $R_1, \dots, R_L$. The free energy is defined as usual by

$$\mathcal{F}(V_1, \dots, V_L) = -\log Z(V_1, \dots, V_L) \tag{2.4}$$

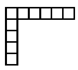
and is understood as a series in traces of V in different representations. We define the generating function $f_{(R_1,\ldots,R_L)}(q,\lambda)$ through the following equation:

$$\mathcal{F}(V) = -\sum_{n=1}^{\infty}\sum_{R_1,\ldots,R_L}\frac{1}{n}f_{(R_1,\ldots,R_L)}(q^n,\mathrm{e}^{-nt})\prod_{\alpha=1}^{L}\mathrm{Tr}_{R_\alpha}V_\alpha^n\,. \tag{2.5}$$

The main result of [25] is that $f_{(R_1,\cdots,R_L)}(q,e^{-t})$ is given by:

$$\begin{aligned} f_{(R_1,\cdots,R_L)}(q,e^{-t}) &= \\ &= (q^{1/2}-q^{-1/2})^{L-2}\sum_{g\geq 0}\sum_{\beta}\sum_{R_1',R_1''\cdots,R_L',R_L''}\prod_{\alpha=1}^{L}c_{R_\alpha\,R_\alpha'\,R_\alpha''}S_{R_\alpha'}(q)\times \\ &\qquad\times N_{(R_1'',\cdots,R_L''),g,\beta}(q^{1/2}-q^{-1/2})^{2g}\mathrm{e}^{-\beta\cdot t}\,. \end{aligned} \tag{2.6}$$

In this formula $R_\alpha, R_\alpha', R_\alpha''$ label representations of the symmetric group S_ℓ, which can be labeled by a Young tableau with a total of ℓ boxes. $c_{R\,R'\,R''}$ are the Clebsch-Gordon coefficients of the symmetric group, and the monomials $S_R(q)$ are defined as follows. If R is a hook representation



(2.7)

with ℓ boxes in total, and with $\ell-d$ boxes in the first row, then

$$S_R(q) = (-1)^d q^{-\frac{\ell-1}{2}+d}\,, \tag{2.8}$$

and it is zero otherwise. Finally, $N_{(R_1,\ldots,R_L),g,\beta}$ are *integers* associated to open string amplitudes. They compute the net number of BPS domain walls of charge β and spin g transforming in the representations R_α of $\mathrm{U}(M)$, where we are using the fact that representations of $\mathrm{U}(M)$ can also be labeled by Young tableaux. It is also useful to introduce a generating functional for these degeneracies as in [25]:

$$\widehat{f}_{(R_1,\cdots,R_L)}(q,e^{-t}) = \sum_{g\geq 0}\sum_{\beta}N_{(R_1,\ldots,R_L),g,\beta}(q^{1/2}-q^{-1/2})^{2g+L-2}\mathrm{e}^{-\beta\cdot t}\,. \tag{2.9}$$

We then have the relation:

$$f_{(R_1,\ldots,R_L)}(q,e^{-t}) = \sum_{R_1',\ldots,R_L'}\prod_{\alpha=1}^{L}M_{R_\alpha R_\alpha'}(q)\widehat{f}_{(R_1,\cdots,R_L)}(q,e^{-t})\,, \tag{2.10}$$

where the matrix $M_{RR'}(q)$ is given by

$$M_{RR'}(q) = \sum_{R''}c_{R\,R'\,R''}S_{R''}(q)$$

and it is symmetric and invertible [25]. The $f_{(R_1,\ldots,R_L)}$ introduced in (2.5) can be extracted from $Z_{(R_1,\ldots,R_L)}$ through a procedure spelled out in detail in [23, 24, 25]. One has, for example,

$$f_{\square\square} = Z_{\square\square} - Z_{\square .}Z_{.\square}\,, \tag{2.11}$$

where . denotes the trivial representation. As it was emphasized in [23, 24, 25], this structure result has interesting consequences for knot theory, since it implies a series of integrality results for knot and link invariants. We will come back to this issue in section 4.

2.2 BPS structure of topological strings on orientifolds

We want to understand now the corresponding BPS structure of closed and open topological string amplitudes on orientifolds without fixed points, like the ones considered in [40, 1]. In [5] the closed case was studied in detail, in the noncompact case, by using large N transitions and the topological vertex. Let us denote by X/I the orientifold obtained by an involution on X. The total free energy has in this case the structure

$$\mathcal{F}\left(X/I,g_s\right)=\frac{1}{2}\mathcal{F}(X,g_s)+\mathcal{F}\left(X/I,g_s\right)_{\rm unor}\ , \tag{2.12}$$

where g_s is the string coupling constant. In the r.h.s. of this equation, the first summand is the contribution of the untwisted sector, and it involves the free energy $\mathcal{F}(X,g_s)$ of the covering X of X/I, after suitably identifying the Kähler classes in the way prescribed by the involution I. This piece of the free energy has an expansion identical to (2.2), but due to the factor 1/2 it involves half-integers instead of integers. The second summand, that we call the unoriented part $\mathcal{F}(X/I,g_s)_{\rm unor}$, is the contribution of the twisted sector, and involves the counting of holomorphic maps from closed non-orientable Riemann surfaces to the orientifold X/I. The Euler characteristic of a closed Riemann surface of genus g and c crosscaps is $\chi=-2g+2-c$ where c is the number of crosscaps. We then have

$$\mathcal{F}(X/I,g_s)_{\rm unor}=\mathcal{F}(X/I,g_s)^{c=1}_{\rm unor}+\mathcal{F}(X/I,g_s)^{c=2}_{\rm unor}, \tag{2.13}$$

which corresponds to the contributions of one and two crosscaps. Following the arguments in [14] we predict the following structure

$$\begin{aligned}\mathcal{F}(X/I,g_s)^{c=1}_{\rm unor}&=\pm\sum_{d\ {\rm odd}}\sum_{g=0}^{\infty}\sum_{\beta}n^{g,c=1}_{\beta}\frac{1}{d}(q^{d/2}-q^{-d/2})^{2g-1}e^{-d\beta\cdot t}\ ,\\ \mathcal{F}(X/I,g_s)^{c=2}_{\rm unor}&=\sum_{d\ {\rm odd}}\sum_{g=0}^{\infty}\sum_{\beta}n^{g,c=2}_{\beta}\frac{1}{d}(q^{d/2}-q^{-d/2})^{2g}e^{-d\beta\cdot t}\ ,\end{aligned} \tag{2.14}$$

where $n^{g,c}_{\beta}$ are integers. The $\pm$ sign in the $c=1$ free energy is due to the two different choices for the sign of the crosscaps, and the restriction to d odd comes, in the case of $c=1$, from the geometric absence of even multicoverings. In the $c=2$ case this was concluded from examination of different examples. The structure results in (2.12), (2.13) and (2.14) were tested in [5] through detailed computations in noncompact geometries.

We now address the generalization to open string amplitudes in orientifolds. We first consider for simplicity the case of a single boundary condition in the orientifold X/I associated to a topological D-brane wrapping a submanifold S. As in the closed string case, the total open string amplitude will have a contribution from untwisted sectors, and a contribution from twisted sectors. We will then write

$$\mathcal{F}(V)=\frac{1}{2}\mathcal{F}_{\rm or}(V)+\mathcal{F}_{\rm unor}(V)\,, \tag{2.15}$$

The contribution from the untwisted sector, $F_{\rm or}(V)$, involves the covering geometry, which will be given by X, the submanifold S, and its image under the involution $I(S)$. In other words, the covering amplitude will involve now *two* different sets of D-branes, in general.

The covering geometry with two sets of branes has the total partition function

$$Z_{\rm cov}(V_1, V_2) = \sum_{R_1, R_2} \mathcal{C}_{R_1 R_2} {\rm Tr}_{R_1} V_1 \, {\rm Tr}_{R_2} V_2 \,, \tag{2.16}$$

where V_1, V_2 are the sources corresponding to S and $I(S)$ and represent open string moduli. Since the two D-branes in S and $I(S)$ are related by an involution, the two-brane amplitude in (2.16) is symmetric under their exchange, i.e. we have

$$\mathcal{C}_{R_1 R_2} = \mathcal{C}_{R_2 R_1} \,. \tag{2.17}$$

In order to obtain $\mathcal{F}_{\rm or}(V)$ we have to make the identification of *both* closed and open string moduli under the involution I. This means identifying the Kähler parameters that appear in $\mathcal{C}_{R_1 R_2}$ (the closed background) but also setting $V_1 = V_2 = V$ (the open background). We then find

$$Z_{\rm or}(V) = \sum_R Z_R^{\rm or} {\rm Tr}_R V \,, \tag{2.18}$$

where

$$Z_R^{\rm or} = \sum_{R_1, R_2} N_{R_1 R_2}^R \mathcal{C}_{R_1 R_2} = \sum_{R'} \mathcal{C}_{R/R' \, R'} \,. \tag{2.19}$$

Here we have used that

$${\rm Tr}_{R_1} V \, {\rm Tr}_{R_2} V = \sum_R N_{R_1 R_2}^R \, {\rm Tr}_R V \tag{2.20}$$

and $N_{R_1 R_2}^R$ are tensor product coefficients. In (2.19) we also used these coefficients to define skew coefficients with labels R/R', as in (A.7). If we denote $\mathcal{C}_R \equiv \mathcal{C}_{R\cdot}$, we have for example

$$Z_{\Box}^{\rm or} = 2\,\mathcal{C}_{\Box} \,, \quad Z_{\Box\!\Box}^{\rm or} = 2\,\mathcal{C}_{\Box\!\Box} + \mathcal{C}_{\Box\,\Box} \,, \quad Z_{\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}}^{\rm or} = 2\mathcal{C}_{\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}} + \mathcal{C}_{\Box\,\Box} \,. \tag{2.21}$$

It turns out that the quantities $Z_R^{\rm or}$ defined in this way have the integrality properties of a one-brane amplitude, as it should. One finds, for example,

$$\widehat{f}_{\Box}^{\rm or} = 2\widehat{f}_{\Box\cdot}^{\rm cov} \,, \quad \widehat{f}_{\Box\!\Box}^{\rm or} = 2\widehat{f}_{\Box\!\Box\cdot}^{\rm cov} - \frac{1}{q^{1/2} - q^{-1/2}} \widehat{f}_{\Box\,\Box}^{\rm cov}, \quad \widehat{f}_{\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}}^{\rm or} = 2\widehat{f}_{\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}\cdot}^{\rm cov} - \frac{1}{q^{1/2} - q^{-1/2}} \widehat{f}_{\Box\,\Box}^{\rm cov}. \tag{2.22}$$

In these equations, the superscript "cov" refers to quantities computed from the two-brane amplitude $\mathcal{C}_{R_1 R_2}$ according to the general rules for open string amplitudes in the usual, oriented case explained above. One can easily verify from the integrality properties of $\widehat{f}_{R_1 R_2}$ as a 2-brane amplitude that indeed $\widehat{f}_R^{\rm or}$ has the integrality properties of a one-brane amplitude. In fact, using the identity

$$\sum_{R', R_1', R_2'} M_{RR'}^{-1} N_{R_1' R_2'}^{R'} M_{R_1' R_1} M_{R_2' R_2} = \frac{1}{q^{-1/2} - q^{1/2}} N_{R_1 R_2}^R \tag{2.23}$$

we can write

$$\widehat{f}_R^{\rm or} = \sum_{R_1 R_2} N_{R_1 R_2}^R \widehat{f}_{R_1 R_2}^{\rm cov} \,, \tag{2.24}$$

where we put $\widehat{f}_{R\cdot} \equiv (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \widehat{f}_R$.

We would like to determine now the structural properties of $\mathcal{F}_{\rm unor}(V)$. This is indeed very easy. The analysis of [25] to determine the structural properties of $F(V)$ in the usual oriented case was based on an analysis of the Hilbert space associated to an oriented Riemann surface $\Sigma_{g,\ell}$ with ℓ holes ending on S and in the relative homology class $\beta \in H_2(X,S)$. The relevant Hilbert space turns out to be

$$\mathrm{Sym}\left(F^{\otimes \ell} \otimes H^*(J_{g,\ell}) \otimes H^*(\mathcal{M}_{g,\ell,\beta})\right), \tag{2.25}$$

where $J_{g,\ell} = \mathbf{T}^{2g+\ell-1}$ is the jacobian of $\Sigma_{g,\ell}$, F is a copy of the fundamental representation of the gauge group, $\mathcal{M}_{g,\ell,\beta}$ is the moduli space of geometric deformations of the Riemann surface inside the Calabi-Yau manifold, and Sym means that we take the completely symmetric piece with respect to permutations of the ℓ holes. Since the bulk of the Riemann surface is not relevant for the action of the permutation group, we can factor out the cohomology of the jacobian $\mathbf{T}^{2g}$. The projection onto the symmetric piece can easily be done using the Clebsch-Gordon coefficients $c_{R\,R'\,R''}$ of the permutation group S_ℓ [12], and one finds

$$\sum_{R\,R'\,R''} c_{R\,R'\,R''} \mathbf{S}_R(F^{\otimes \ell}) \otimes \mathbf{S}_{R'}(H^*((\mathbf{S}^1)^{\ell-1})) \otimes \mathbf{S}_{R''}(H^*(\mathcal{M}_{g,\ell,\beta}))\,, \tag{2.26}$$

where $\mathbf{S}_R$ is the Schur functor that projects onto the corresponding subspace. The space $\mathbf{S}_R(F^{\otimes\ell})$ is nothing but the vector space underlying the irreducible representation R of $\mathrm{U}(M)$. $\mathbf{S}_{R'}(H^*((\mathbf{S}^1)^{\ell-1}))$ gives the hook Young tableau, and the Euler characteristic of $\mathbf{S}_{R''}(H^*(\mathcal{M}_{g,\ell,\beta}))$ is the integer invariant $N_{R'',g,\beta}$. Therefore, the above decomposition corresponds very precisely to (2.6) (here we are considering for simplicity the one-brane case).

In the case of an *unoriented* Riemann surface, the above argument goes through, with the only difference that now the jacobian is $J_{g,c,\ell} = \mathbf{T}^{2g-1+\ell+c}$, where $c = 1, 2$ denotes the number of crosscaps. Therefore, the analysis of the cohomology associated to the boundary is the same. We then conclude that

$$\mathcal{F}_{\rm unor}(V) = -\sum_R \sum_{d\,{\rm odd}} \frac{1}{d} f_R^{\rm unor}(q^d, e^{-dt}) \mathrm{Tr}_R V^d\,, \tag{2.27}$$

and using again (2.10) one can obtain new functions

$$\hat{f}_R^{\rm unor} = \sum_R{}' M^{-1}_{RR'} f_{R'}^{\rm unor} \tag{2.28}$$

with contributions from one and two crosscaps:

$$\hat{f}_R^{\rm unor} = \hat{f}_R^{c=1} + (q^{1/2} - q^{-1/2}) \hat{f}_R^{c=2}\,, \tag{2.29}$$

and we finally have

$$\hat{f}_R^c(q, e^{-t}) = \sum_{g,\beta} N^c_{R,g,\beta} (q^{1/2} - q^{-1/2})^{2g} e^{-\beta\cdot t}\,. \tag{2.30}$$

Each crosscap contributes then an extra factor of $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, as in the closed case.

In real life, what one computes is the total amplitude in the l.h.s. of (2.15), in terms of

$$\mathcal{F}(V) = -\log Z(V) = -\log\left(\sum_R Z_R \mathrm{Tr}_R\, V\right), \tag{2.31}$$

and one wants to find the unoriented contribution to the amplitude after subtracting the oriented contribution. The above formulae give a precise prescription to compute f_R^{unor}. The results one finds, up to three boxes, are the following:

$$\begin{aligned}
f^{\mathrm{unor}}_{\yng(1)} &= Z_{\yng(1)} - \mathcal{C}_{\yng(1)}\,,\\
f^{\mathrm{unor}}_{\yng(2)} &= Z_{\yng(2)} - \frac{1}{2}Z^2_{\yng(1)} - \mathcal{C}_{\yng(2)} + \frac{1}{2}\mathcal{C}^2_{\yng(1)} - \frac{1}{2}f^{\mathrm{cov}}_{\yng(1)\,\yng(1)}\,,\\
f^{\mathrm{unor}}_{\yng(1,1)} &= Z_{\yng(1,1)} - \frac{1}{2}Z^2_{\yng(1)} - \mathcal{C}_{\yng(1,1)} + \frac{1}{2}\mathcal{C}^2_{\yng(1)} - \frac{1}{2}f^{\mathrm{cov}}_{\yng(1)\,\yng(1)}\,,\\
f^{\mathrm{unor}}_{\yng(3)} &= Z_{\yng(3)} - Z_{\yng(2)}Z_{\yng(1)} + \frac{1}{3}Z^3_{\yng(1)} - \mathcal{C}_{\yng(3)} + \mathcal{C}_{\yng(2)}\mathcal{C}_{\yng(1)} - \frac{1}{3}\mathcal{C}^3_{\yng(1)} - \frac{1}{3}f^{\mathrm{unor}}_{\yng(1)}(q^3,Q^3) - f^{\mathrm{cov}}_{\yng(2)\,\yng(1)}\,,
\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}
f^{\mathrm{unor}}_{\yng(2,1)} &= Z_{\yng(2,1)} - Z_{\yng(2)}Z_{\yng(1)} - Z_{\yng(1,1)}Z_{\yng(1)} + \frac{2}{3}Z^3_{\yng(1)} - \mathcal{C}_{\yng(2,1)} + \mathcal{C}_{\yng(2)}\mathcal{C}_{\yng(1)} + \mathcal{C}_{\yng(1,1)}\mathcal{C}_{\yng(1)} + \frac{2}{3}\mathcal{C}^3_{\yng(1)} +\\
&\quad + \frac{1}{3}f^{\mathrm{unor}}_{\yng(1)}(q^3,Q^3) - \frac{1}{2}\left(f^{\mathrm{cov}}_{\yng(2)\,\yng(1)} + f^{\mathrm{cov}}_{\yng(1,1)\,\yng(1)}\right),\\
f^{\mathrm{unor}}_{\yng(1,1,1)} &= Z_{\yng(1,1,1)} - Z_{\yng(1,1)}Z_{\yng(1)} + \frac{1}{3}Z^3_{\yng(1)} - \mathcal{C}_{\yng(1,1,1)} - \mathcal{C}_{\yng(1,1)}\mathcal{C}_{\yng(1)} + \frac{1}{3}\mathcal{C}^3_{\yng(1)} -\\
&\quad - \frac{1}{3}f^{\mathrm{unor}}_{\yng(1)}(q^3,Q^3) - f^{\mathrm{cov}}_{\yng(1,1)\,\yng(1)}\,.
\end{aligned} \tag{2.33}$$

The above considerations are easily extended to the case in which we have L sets of D-branes in the orientifold geometry. The covering amplitude involves now $2L$ D-branes, and reads

$$Z_{\mathrm{cov}} = \sum_{R_1,S_1,\cdots,R_L,S_L} \mathcal{C}_{R_1S_1\cdots R_LS_L} \mathrm{Tr}_{R_1}V_1 \mathrm{Tr}_{S_1}W_1 \cdots \mathrm{Tr}_{R_L}V_1 \mathrm{Tr}_{S_L}W_L\,. \tag{2.34}$$

The oriented amplitude is obtained by identifying the moduli in pairs under I, and is given by

$$Z^{\mathrm{or}}_{Q_1\cdots Q_L} = \sum_{R_i,S_i} N^{Q_1}_{R_1S_1}\cdots N^{Q_L}_{R_LS_L}\mathcal{C}_{R_1S_1\cdots R_LS_L}\,. \tag{2.35}$$

The equations (2.15), (2.27) and (2.30) generalize in an obvious way, but now we have

$$\widehat{f}^{c}_{(R_1\cdots R_L)}(q,e^{-t}) = \sum_{g,\beta} N^{c}_{(R_1,\cdots,R_L),g,\beta}(q^{1/2}-q^{-1/2})^{2g+L-1}e^{-\beta\cdot t}\,, \tag{2.36}$$

where the extra $L-1$ factors of $q^{1/2}-q^{-1/2}$ have the same origin as in (2.9).

3. Examples of open string amplitudes

In this section we study in detail some examples and verify the above formulae for the unoriented part of the free energy. In order to do that, we have to compute the total amplitudes Z_R in orientifold geometries. These amplitudes can be obtained in three ways:

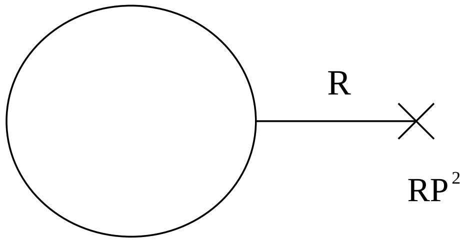


Figure 1: Toric diagram for the quotient X/I of a local, toric Calabi-Yau manifold with a single $\mathbb{RP}^2$.

by using the unoriented localization methods of [9, 5], by using mirror symmetry [1], and by using Chern-Simons theory and the topological vertex. For the examples of open string amplitudes studied in this section we will use the topological vertex of [2], which can be adapted to the orientifold case [5], and also localization. We first summarize very briefly the results of [5] on the topological vertex on orientifolds, and then we study in detail three examples. Finally we check some of the topological vertex results with unoriented localization.

3.1 The topological vertex on orientifolds

Let us consider a quotient X/I of a local, toric Calabi-Yau manifold X by an involution I without fixed points which can be represented as in figure 1. We have a bulk geometry, represented by the blob, attached to an $\mathbb{RP}^2$ through an edge with representation S. Let us denote by $\mathcal{O}_S$ the amplitude for the blob with the external leg. In [5] we proposed the following formula for the total partition function:

$$Z = \sum_{S=S^T} \mathcal{O}_S Q^{\ell(S)/2} (-1)^{\frac{1}{2}(\ell(S)\mp r(S))} \tag{3.1}$$

where the sum is over all self-conjugate representations S. Here $r(S)$ denotes the rank of S, Q is the exponentiated Kähler parameter corresponding to the $\mathbb{RP}^2$, and the $\mp$ sign is correlated with the choice of $\pm$ sign for the crosscaps. In [5] the above prescription was used to compute closed string amplitudes, and we conjectured the following identity:

$$\begin{aligned}\frac{1}{S_{00}^{\mathrm{SO}(N)/\,\mathrm{Sp}(N)}} \sum_{R=R^T} C_{R_1 R_2^T R} Q^{\ell(R)/2} (-1)^{\frac{1}{2}(\ell(R)\mp r(R))} &= \\ = q^{-\frac{\kappa_{R_2}}{2}} Q^{\frac{1}{2}(\ell(R_1)+\ell(R_2))} \mathcal{W}_{R_1 R_2}^{\mathrm{SO}(N)/\,\mathrm{Sp}(N)}&,\end{aligned} \tag{3.2}$$

where $C_{R_2^T R R_1}$ is the topological vertex of [2], $\mathcal{W}_{R_1 R_2}^{\mathrm{SO}(N)/\,\mathrm{Sp}(N)}$ is the SO/Sp Chern-Simons expectation value of the Hopf link with linking number $+1$ (after setting the Chern-Simons variable λ defined in (4.4) to be $\lambda = Q^{-1}$), and $S_{00}^{\mathrm{SO}(N)/\,\mathrm{Sp}(N)}$ is the partition function of SO/Sp Chern-Simons theory on $\mathbf{S}^3$. We refer the reader to [31, 5] for explicit formulae for the Hopf link invariants. The identity (3.2) was used in [5] to show that the results obtained with (3.1) coincide with those that are obtained from large N transitions involving SO/Sp gauge groups. In the examples that follow we will use (3.1) to compute open string amplitudes on orientifolds, making use as well of the identity (3.2). We provide a proof of (3.2) in appendix B.

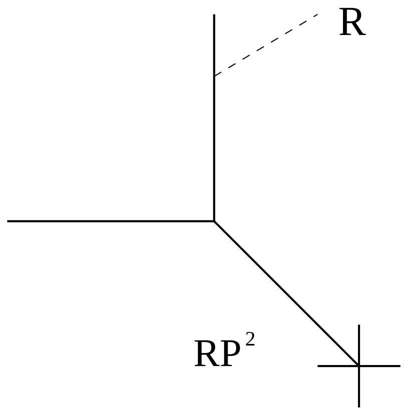


Figure 2: A D-brane in an outer leg of the orientifold of the resolved conifold.

3.2 The SO/Sp framed unknot

We start with the simplest nontrivial Calabi-Yau orientifold, namely the orientifold of the resolved conifold. The resolved conifold X is a noncompact Calabi-Yau threefold which admits a toric description given by the following toric data:

$$\begin{array}{ccccc} & X_1 & X_2 & X_3 & X_4 \\ \mathbf{C}^* & 1 & 1 & -1 & -1 \end{array} \tag{3.3}$$

Therefore, it is defined as the space obtained from

$$|X_1|^2 + |X_2|^2 - |X_3|^2 - |X_4|^2 = t \tag{3.4}$$

after quotienting by the U(1) action specified by the charges in (3.3). The involution

$$I : (X_1, X_2, X_3, X_4) \to (\overline{X}_2, -\overline{X}_1, \overline{X}_4, -\overline{X}_3)\,. \tag{3.5}$$

leads to an orientifold model whose target X/I contains a single $\mathbb{RP}^2$ obtained from the quotient of the $\mathbb{P}^1$ of X by I. This geometry was first considered in [40, 1] and further studied in [5].

Let us now put a D-brane in an *outer* leg of the orientifold geometry. In the oriented case, the open string amplitude labelled by R is computed by the Chern-Simons invariant of the framed unknot with gauge group $\mathrm{U}(N)$ (see for example [31, 32]). We want to study now the unoriented case. In order to extract the unoriented string amplitudes, we have to compute both the total amplitudes Z_R and the covering amplitudes $\mathcal{C}_{R_1R_2}$. Let us start analyzing the covering amplitude.

The covering geometry involves both the original D-brane and its image under the involution I, and a simple analysis shows that we have to consider *two* D-branes in opposite legs as qdepicted in figure 3. The amplitude for this two-brane configuration can be computed by using the topological vertex of [2] (see appendix A for a list of useful formulae and properties of the vertex). A simple application of the rules in [2] gives

$$\mathcal{C}_{R_1R_2} = \frac{1}{Z_{\mathbb{P}^1}} \sum_R C_{R_1R\cdot} C_{R^T\cdot R_2} (-Q)^{\ell(R)} = \frac{1}{Z_{\mathbb{P}^1}} \sum_R W_{R_1R^T} W_{RR_2} (-Q)^{\ell(R)}\,, \tag{3.6}$$

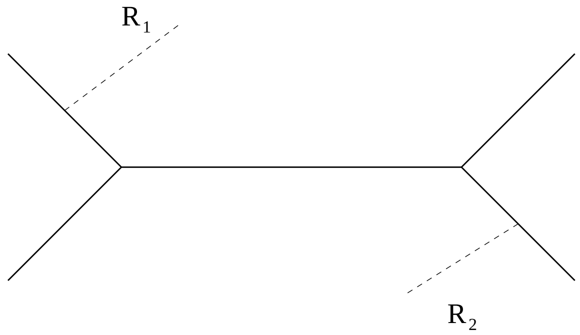


Figure 3: The covering configuration contains two D-branes.

where $Q = e^{-t}$, $Z_{\mathbb{P}^1}$ is the partition function of the resolved conifold

$$Z_{\mathbb{P}^1} = \prod_{k=1}^{\infty} (1 - Qq^k)^k \,, \tag{3.7}$$

and the quantities $W_{R_1 R_2}$ are defined in (A.4). The above quotient of series can be computed in a closed way by using the techniques of [17, 10], and in fact one obtains two equivalent expressions. The first expression is

$$\mathcal{C}_{R_1 R_2} = W_{R_1} W_{R_2} \prod_k (1 - q^k Q)^{C_k(R_1, R_2)} \,, \tag{3.8}$$

where the coefficients $C_k(R_1, R_2)$ are given by (A.13) or (A.14). Notice that (3.8) is a Laurent polynomial in $q^{\pm\frac{1}{2}}$ and a *polynomial* in Q. There is, however, a second expression for $\mathcal{C}_{R_1 R_2}$ which involves skew quantum dimensions. The derivation uses the representation of the vertex in terms of skew Schur functions given in (A.6). Define first:

$$\mathcal{W}_R = (\dim_q^{\mathrm{U}(N)} R)(\lambda = Q^{-1}) \,, \tag{3.9}$$

where λ is again the Chern-Simons variable (4.4), and the quantum dimension is defined in (A.2). Then we have, after using (A.10),

$$\mathcal{C}_{R_1 R_2} = q^{\frac{\kappa_{R_1} + \kappa_{R_2}}{2}} Q^{\frac{\ell(R_1) + \ell(R_2)}{2}} \sum_R (-1)^{\ell(R)} \mathcal{W}_{R_1^T/R} \mathcal{W}_{R_2^T/R^T} \,. \tag{3.10}$$

We now compute the total amplitude for the configuration depicted in figure 2. To do this we can use (3.1), where $\mathcal{O}_S$ is now an open string amplitude given by $C_{\cdot RS}$. One finds

$$\begin{aligned} Z_R &= \frac{1}{Z_{X/I}} \sum_{R' = R'^T} C_{\cdot R R'} Q^{\ell(R')/2} (-1)^{\frac{1}{2}(\ell(R') \mp r(R'))} \\ &= q^{\frac{\kappa_R}{2}} \mathcal{W}_{R^T}^{\mathrm{SO}\,/\,\mathrm{Sp}} \,, \end{aligned} \tag{3.11}$$

where we have used the formula (3.2) to express the amplitude in terms of $SO/\,\mathrm{Sp}$ quantum dimensions. We then see that the *total* brane amplitude in figure 2 is given by the Chern-Simons invariant of an unknot for gauge groups $SO/\,\mathrm{Sp}$. To obtain the unoriented piece of this amplitude, we have to subtract the covering contribution, which involves a nontrivial combination of quantum dimensions for $\mathrm{U}(N)$. For a *framed* D-brane one should simply change

$$
\begin{aligned}
Z_R &\to (-1)^{\ell(R)p} q^{\frac{p\kappa_R}{2}} Z_R \,, \\
\mathcal{C}_{R_1 R_2} &\to (-1)^{(\ell(R_1)+\ell(R_2))p} q^{p\frac{\kappa_{R_1}+\kappa_{R_2}}{2}} \mathcal{C}_{R_1 R_2} \,,
\end{aligned}
\tag{3.12}
$$

since in the covering configuration one has to put the same framing in both legs, by symmetry.

We can now compute f_R^{unor} by using the results of the previous section. We will present explicit results only up to three boxes. The first thing one finds is that $f_R^{c=2}$ vanishes at this order in R. For $f_R^{c=1}$ one finds (we present here the results for $\mathrm{SO}(N)$; for $\mathrm{Sp}(N)$ one only has to change the overall sign of the $c=1$ contributions):

$$
\begin{aligned}
\widehat{f}^{c=1}_{\square} &= (-1)^p Q^{1/2} \,, \\
\widehat{f}^{c=1}_{\square\square} &= \frac{q^{1-p}\left(1 - q^p - q^{1+p} + q^{1+2p}\right) Q^{1/2}\,(-1+Q)}{(q-1)^2(q+1)} \,, \\
\widehat{f}^{c=1}_{\substack{\square\\\square}} &= q^{-p}\frac{\left(1 - q^{1+p} - q^{2+p} + q^{3+2p}\right) Q^{1/2}\,(-1+Q)}{(q-1)^2(q+1)} \,,
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
\widehat{f}^{c=1}_{\square\square\square} = {} & \frac{(-1)^p q^{2-3p}(-1+q^p)(-1+q^{1+p})Q^{1/2}(-1+Q)}{(-1+q)^4(1+q)^2(1+q^2+q^4)} \times \\
& \times [-q + q^{2p} + q^{1+p}(-1-q+2q^p-q^{2p}) + q^{2(1+p)}(2+q-q^p-q^{2p}) + \\
& \quad + Q(1+q^p-q^{2p}+q^{1+p}(1-2q^p)+q^{2(1+p)}(-2-q+q^p)+q^{3(1+p)}(1+q^p))] \,, \\
\widehat{f}^{c=1}_{\substack{\square\square\\\square}} = {} & \frac{(-1)^p q^{-3p}(-1+q^{1+p})Q^{1/2}}{(-1+q)^4(1+q)^2(1+q^2+q^4)} \times \\
& \times [q(-1+q^p)(1+q^p+q^{1+p}(1-2q^p)+q^{2(1+p)}(-2-2q)+q^{3(1+p)}(1+q)+q^{4(1+p)}) - \\
& \quad - Q(1+q)(-1+q^{1+p}(-1+3q^p)+q^{2(1+p)}(2+3q-3q^p) + \\
& \quad + q^{3(1+p)}(-2-3q^2)+q^{4(1+p)}+q^{5(1+p)}) + \\
& \quad + Q^2(-1+q^{1+p}(-1+2q^p)+q^{2(1+p)}(2+3q+q^2-q^p) + \\
& \quad + q^{3(1+p)}(-3-2q-2q^2)+q^{5+4p}+q^{6+5p})] \,, \\
\widehat{f}^{c=1}_{\substack{\square\\\square\\\square}} = {} & \frac{(-1)^p q^{-1-3p}(1-q^{1+p}-q^{2+p}+q^{3+2p})Q^{1/2}(-1+Q)}{(-1+q)^4(1+q)^2(1+q^2+q^4)} \times \\
& \times [-q(1+q^{1+p}(1+q-q^p)+q^{2(1+p)}(-2-2q-q^2)+q^{3(1+p)}(1+q)+q^{5+4p})) + \\
& \quad + Q(1+q^{1+p}(1+q)+q^{2(1+p)}(-1-2q-2q^2-q^3)+q^{3(1+p)}(q^2+q^3)+q^{7+4p})] \,.
\end{aligned}
$$

One can indeed check that, for any integer p, the above polynomials are of the form predicted in (2.30) (they are polynomials in $(q^{1/2}-q^{-1/2})^2$ with integer coefficients).

3.3 $\mathbb{P}^2$ attached to $\mathbb{RP}^2$

The next example we consider is the orientifold studied in [9, 5], with a D-brane located in an outer leg. In this case, the covering space consists of two $\mathbb{P}^2$'s connected by a $\mathbb{P}^1$, with two D-branes in opposite legs (the geometry is shown in figure 4). Let us now define the

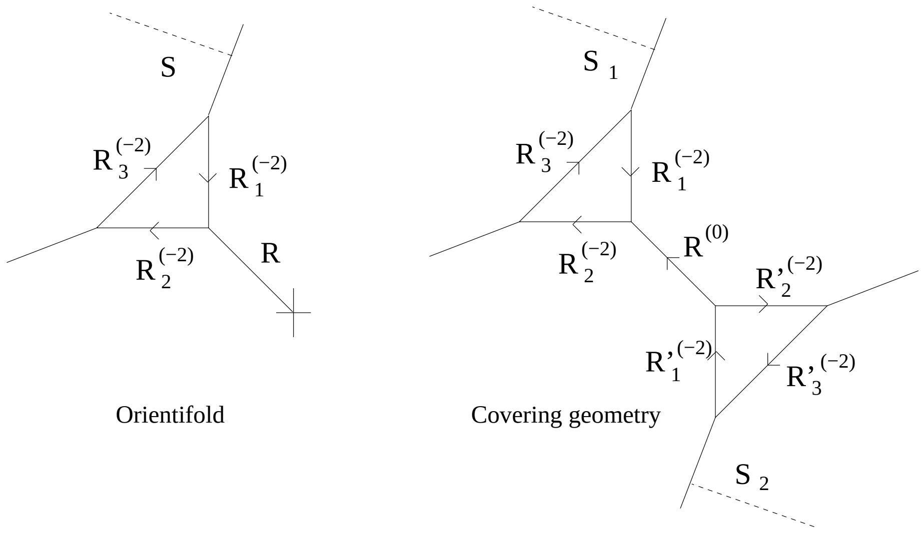


Figure 4: A D-brane in an outer leg of the orientifold of the two $\mathbb{P}^2$'s connected by a $\mathbb{P}^1$.

following operator corresponding to the $\mathbb{P}^2$ with an outer D-brane:

$$\mathcal{O}_{RS} = \sum_{R_i} q^{\sum_i \kappa_{R_i}} (-1)^{\sum_i \ell(R_i)} C_{S R_3 R_1^T} C_{\cdot R_2 R_3^T} C_{R_1 R_2^T R} e^{-t \sum_i \ell(R_i)} ,$$

where S is the representation attached to the D-brane. Using the topological vertex rules we can write, for arbitrary framing p (Z_{closed} is the amplitude without D-branes):

$$\mathcal{C}_{S_1 S_2} = \frac{1}{Z^{\text{cov}}_{\text{closed}}} \sum_R (-1)^{p(\ell(S_1)+\ell(S_2))} q^{\frac{p}{2}(\kappa_{S_1}+\kappa_{S_2})} \mathcal{O}_{RS_1} \mathcal{O}_{R^T S_2} (-Q)^{\ell(R)}$$

and

$$Z^{\text{cov}}_{\text{closed}} = 1 + \sum_R \mathcal{O}_{R\cdot} \mathcal{O}_{R^T\cdot} (-Q)^{\ell(R)} , \tag{3.14}$$

where in the last equation we have singled out the term where all the representations are trivial. As we do not have a closed expression for $\mathcal{C}_{S_1 S_2}$ we have to evaluate Z_{or} order by order in Q and e^{-t}.

Let us compute now Z_S by using the topological vertex rules for orientifolds developed in [5]. We find that

$$Z_S = \frac{1}{Z_{\text{closed}}} (-1)^{p\ell(S)} q^{\frac{p\kappa_S}{2}} \sum_{R=R^T} \mathcal{O}_{RS} Q^{\ell(R)/2} (-1)^{\frac{1}{2}(\ell(R)\mp r(R))} ,$$

where

$$Z_{\text{closed}} = 1 + \sum_{R=R^T} \mathcal{O}_{R\cdot} Q^{\ell(R)/2} (-1)^{\frac{1}{2}(\ell(R)\mp r(R))} . \tag{3.15}$$

Using the results in the previous section, we can compute the functions $\widehat{f}^{\,c}_S(q, Q, e^{-t})$. We find the following results at low order, for arbitrary framing p (again we present the results

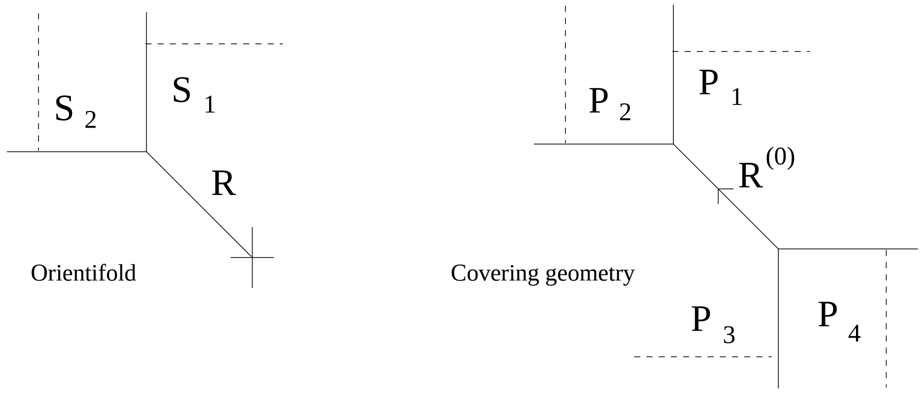


Figure 5: Two adjacent D-branes in the outer legs of the orientifold of the resolved conifold.

for SO(N)):

$$
\begin{aligned}
\widehat{f}^{c=1}_{\square} &= (-1)^p \Big[-Q^{1/2}e^{-t} + 4Q^{1/2}e^{-2t} - (35+8\,z)Q^{1/2}e^{-3t} + \\
&\qquad + (400+344z+112z^2+13z^3)Q^{1/2}e^{-4t} - 2Q^{3/2}e^{-2t} + (30+6z)Q^{3/2}e^{-3t} - \\
&\qquad - (488+359\,z+104\,z^2+11\,z^3)Q^{3/2}e^{-4t} - 3\,Q^{5/2}e^{-3t} + \\
&\qquad + (132+59\,z+8\,z^2)Q^{5/2}e^{-4t} + \cdots \Big], \\
\widehat{f}^{c=2}_{\square} &= -(-1)^p \Big[Q^2e^{-3t} - (15+7\,z+z^2)Q^2e^{-4t} + 2\,Q^4e^{-4t} + \cdots \Big], \\
\widehat{f}^{c=1}_{\square\!\square} &= \frac{q^{-p+1}(-1+q^p)(-q+q^p)}{(q-1)^2(q+1)}Q^{1/2}e^{-t} - \frac{3q^{-p+1}(-1+q^p)^2}{(q-1)^2}Q^{1/2}e^{-2t} + \cdots\,, \\
\widehat{f}^{c=2}_{\square\!\square} &= \frac{q^{-p+1}(q^p-1)^2}{(q-1)^2}Q^2e^{-3t} + \cdots\,, \\
\widehat{f}^{c=1}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= \frac{q^{-p+1}(q^p-1)(q^{1+p}-1)}{(q-1)^2(q+1)}Q^{1/2}e^{-t} - \frac{3q^{-p}(q^{1+p}-1)^2}{(q-1)^2}Q^{1/2}e^{-2t} + \cdots\,, \\
\widehat{f}^{c=2}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= \frac{q^{-p}(q^{1+p}-1)^2}{(q-1)^2}Q^2e^{-3t} + \cdots\,,
\end{aligned}
$$

where $z \equiv (q^{1/2} - q^{-1/2})^2$.

By comparing with (2.30), it is easy to see that the results above have the expected polynomial form with integer coefficients for any p. In contrast to the example above of a D-brane in the orientifold of the conifold, in this example there are nonzero amplitudes with an even number of crosscaps.

3.4 SO/Sp Hopf link invariant

Our third and final example is the orientifold of the resolved conifold with two adjacent D-branes in the outer legs. The covering geometry now involves four sets of D-branes in the outer legs of the resolved conifold, oppositely identified by the involution. The geometry is shown in figure 5.

Using the topological vertex, we find for the covering amplitude (for arbitrary framings p_1 and p_2):

$$\begin{aligned}\mathcal{C}_{P_1P_2P_3P_4} &= \frac{1}{Z_{\mathbb{P}^1}}\sum_R (-1)^{p_1(\ell(P_1)+\ell(P_3))+p_2(\ell(P_2)+\ell(P_4))}\times \\ &\quad\times q^{\frac{p_1}{2}(\kappa_{P_1}+\kappa_{P_2})+\frac{p_2}{2}(\kappa_{P_2}+\kappa_{P_4})}(-Q)^{\ell(R)}C_{R^TP_1P_2}C_{RP_3P_4}\,. \end{aligned} \tag{3.16}$$

To obtain the oriented amplitude from (3.16) we have to identify the moduli P_1 (P_2) with P_3 (P_4) as explained in (2.35). We can rewrite (3.16) by using the expression of the topological vertex in terms of Schur functions (A.6):

$$\begin{aligned}\mathcal{C}_{P_1P_2P_3P_4} &= \frac{1}{Z_{\mathbb{P}^1}}q^{\frac{1}{2}(\sum_{i=1}^4\kappa_{R_i})}s_{R_1^T}(q^\rho)s_{R_3^T}(q^\rho)\times \\ &\quad\times\sum_{\eta_1,\eta_2}(-Q)^{\ell(\eta_1)}s_{R_2^T/\eta_1}(q^{\ell(R_1)+\rho})s_{R_4^T/\eta_2}(q^{\ell(R_3)+\rho})\times \\ &\quad\times\sum_R s_{R^T/\eta_1}(-Qq^{\ell(R_1^T)+\rho})s_{R/\eta_2}(q^{\ell(R_3^T)+\rho})\,. \end{aligned} \tag{3.17}$$

By using the identities (A.9), (A.10) and (A.15) we finally obtain that

$$\begin{aligned}\mathcal{C}_{P_1P_2P_3P_4} &= q^{\frac{1}{2}(\sum_{i=1}^4\kappa_{R_i})}s_{R_1^T}(q^\rho)s_{R_3^T}(q^\rho)\prod_k(1-Qq^k)^{C_k(R_1^T,R_3^T)}\times \\ &\quad\times\sum_\eta(-Q)^{\ell(\eta)}s_{R_2^T/\eta^T}(q^{\ell(R_1)+\rho},Qq^{-\ell(R_3^T)-\rho})s_{R_4^T/\eta}(q^{\ell(R_3)+\rho},Qq^{-\ell(R_1^T)-\rho})\,, \end{aligned} \tag{3.18}$$

where we defined the functions

$$s_{R/Q}(x,y)=\sum_\eta s_{R/\eta}(x)s_{\eta/Q}(y)\,, \tag{3.19}$$

and the coefficients $C_k(R_1^T,R_3^T)$ are defined in (A.13) or (A.14). Notice that (3.18) is a polynomial in Q.

Now that we have our final expression for the covering amplitude, let us look at the full amplitude. The vertex rules for orientifolds developed in our previous paper [5] tell us that, for the amplitude where there are two D-branes in the outer legs, one has (for arbitrary framings p_1 and p_2)

$$\begin{aligned}Z_{S_1S_2} &= \frac{1}{Z_{X/I}}\sum_{R=R^T}(-1)^{p_1\ell(S_1)+p_2\ell(S_2)}q^{\frac{1}{2}(p_1\kappa_{S_1}+p_2\kappa_{S_2})}C_{S_1S_2R}Q^{\ell(R)/2}(-1)^{\frac{1}{2}(\ell(R)\mp r(R))} \\ &= (-1)^{p_1\ell(S_1)+p_2\ell(S_2)}q^{\frac{1}{2}(p_1\kappa_{S_1}+p_2\kappa_{S_2})}q^{\frac{1}{2}\kappa_{S_2}}Q^{\frac{1}{2}(\ell(S_1)+\ell(S_2))}\mathcal{W}^{\mathrm{SO}(N)/\,\mathrm{Sp}(N)}_{S_1S_2^t}\,, \end{aligned} \tag{3.20}$$

where we used again (3.2). This time, we see that the *total* amplitude of the two D-brane configuration in the orientifold of the conifold is given by the $SO/$ Sp Chern-Simons invariants of the Hopf link.

By substracting the oriented piece from the unoriented amplitude, and using the results of the previous section, we can compute the $N^c_{(S_1,S_2),g,\beta}$ integer invariants through the $\widehat{f}^c_{S_1S_2}$ functions. As noted in (2.9) we now expect a slightly different structure for the $\widehat{f}^c_{S_1S_2}$

functions than the one given by (2.30), since $L = 2$. Namely, we expect

$$\widehat{f}^{c}_{S_1 S_2} = \sum_{g,\beta} N^{c}_{(S_1,S_2),g,\beta} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g+1} Q^{\beta} \,. \tag{3.21}$$

with the $\widehat{f}^{c}_{S_1 S_2}$ functions defined as in (2.29).

We obtain the following results for SO(N):

$$\begin{aligned}
\widehat{f}^{c=1}_{\square\,\square} &= (-1)^{p_1+p_2} Q^{1/2} (1-Q)(q^{1/2} - q^{-1/2}) \,, \\
\widehat{f}^{c=1}_{\square\!\square\,\square} &= \frac{(-1)^{2p_1+p_2} Q^{1/2}(1-Q) q^{-p_1-1/2}}{q-1} (-q + 2q^{1+p_1} - q^{1+2p_1} + Q(q^2 + q^{2p_1} - 2q^{1+p_1})) \,, \\
\widehat{f}^{c=1}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}\,\square} &= \frac{(-1)^{2p_1+p_2} Q^{1/2}(1-Q) q^{-p_1-1/2}}{q-1} (-(-1+q^{1+p_1})^2 + Qq(-1+q^{p_1})^2) \,.
\end{aligned} \tag{3.22}$$

It is straigthforward to show that for any fixed framings p_1 and p_2 the $\widehat{f}$ functions (3.22) have the structure predicted by (3.21) with integer invariants $N^{c}_{(S_1,S_2),g,\beta}$. Up to the order $\ell(S_1) + \ell(S_2) = 3$ the contributions with two crosscaps vanish.

3.5 Localization computations

In the previous subsections we found many open BPS invariants using the topological vertex prescription of [5] and the structure predictions of section 2. As far as we are aware these invariants have never been computed before. Therefore it would be nice to have an independent check of our results which does not rely on large N duality.

In [9, 5] localization techniques were defined to compute closed unoriented Gromov-Witten invariants of Calabi-Yau orientifolds. In this section we will extend these techniques to the case of *open* unoriented Gromov-Witten invariants, therefore providing an alternative and independent way to compute the invariants of the previous subsections.

In order to compare our results with localization computations we have to extract open Gromov-Witten invariants from the f polynomials. First let us recall the definition of the f functions (2.5):

$$\mathcal{F}^c(V_1,\ldots,V_L) = -\sum_{d=1}^{\infty} \sum_{R_1,\cdots,R_L} \frac{1}{d} f^{c}_{(R_1,\cdots,R_L)}(q^d, \mathrm{e}^{-dt}) \prod_{\alpha=1}^{L} \mathrm{Tr}_{R_\alpha} V_\alpha^d \,, \tag{3.23}$$

where we added the superscript c for the number of crosscaps. As usual, we can also work in the $\vec{k}$ basis. In this basis the free energy reads (see [24]):

$$\mathcal{F}^c(V_1,\ldots,V_L) = -\sum_{\{\vec{k}^{\alpha}\}} W^{(\mathrm{conn}),c}_{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})} \prod_{\alpha}^{L} \frac{1}{z_{\vec{k}^{(\alpha)}}} \Upsilon_{\vec{k}^{(\alpha)}}(V_\alpha) \,, \tag{3.24}$$

where we defined the connected vevs $W^{(\mathrm{conn}),c}_{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}$, and $z_{\vec{k}} = \prod_m k_m! m^{k_m}$. Since $q = e^{ig_s}$, we can expand the r.h.s of (3.24) in g_s. We find a series with the structure [24]:

$$\begin{aligned}
\mathcal{F}^c(V_1,\ldots,V_L) &= \sum_{g=0}^{\infty} i^{\sum_{\alpha=1}^{L} |\vec{k}^{(\alpha)}|+c} F^{c}_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})} g_s^{2g-2+c+\sum_{\alpha=1}^{L} |\vec{k}^{(\alpha)}|} \Upsilon_{\vec{k}^{(\alpha)}}(V_\alpha) \\
&= -\left(\prod_{\alpha}^{L} \frac{1}{z_{\vec{k}^{(\alpha)}}} \right) W^{(\mathrm{conn}),c}_{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})} \Upsilon_{\vec{k}^{(\alpha)}}(V_\alpha) \,,
\end{aligned} \tag{3.25}$$

where $F^c_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}$ is the generating functional for open Gromov-Witten invariants at genus g, with c crosscaps and fixed boundary conditions given by $(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})$. The factor of $i^{\sum_{\alpha=1}^{L}|\vec{k}^{(\alpha)}|+c}$ is necessary to compare Chern-Simons (or topological vertex) results with localization computations [32]. Thus, we see that to extract open Gromov-Witten invariants we have to compute the connected vevs $W^{(\mathrm{conn}),c}_{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}$ from the f functions. Such a relation has been deduced in [24]:

$$W^{(\mathrm{conn}),c}_{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})} = \sum_{d|\vec{k}^{(\alpha)},\ d\ \mathrm{odd}} d^{\sum_\alpha|\vec{k}^{(\alpha)}|-1} \sum_{\{R_\alpha\}} \prod_{\alpha=1}^{L} \chi_{R_\alpha}(C(\vec{k}^{(\alpha)}_{1/d})) f^c_{(R_1,\ldots,R_L)}(q^d,\mathrm{e}^{-dt}), \tag{3.26}$$

where $C(\vec{k})$ is the conjugacy class associated to a vector $\vec{k}$, which has k_j cycles of length j, and χ_R is the character of the symmetric group S_ℓ. In (3.26) the vector $\vec{k}_{1/d}$ is defined as follows. Fix a vector $\vec{k}$, and consider all the positive integers d that satisfy the following condition: $d|j$ for every j with $k_j \neq 0$. When this happens, we will say that "d divides $\vec{k}$", and we will denote this as $d|\vec{k}$. We can then define the vector $\vec{k}_{1/d}$ whose components are $(\vec{k}_{1/d})_i = k_{di}$. In (3.26) the integer d has to divide all the vectors $\vec{k}^{(\alpha)}$, $\alpha = 1,\ldots,L$. Note that in (3.26) the sum is only over d *odd*: this is because in the unoriented case only odd multicovers contribute.

Using (3.25) and (3.26) one can find expressions for the generating functionals of open Gromov-Witten invariants in terms of f functions. Let us define the all genera generating functionals for open Gromov-Witten invariants with c crosscaps and fixed boundary conditions given by $(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})$:

$$F^c_{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})} = \sum_{g=0}^{\infty} F^c_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})} g_s^{2g-2+c+\sum_{\alpha=1}^{L}|\vec{k}^{(\alpha)}|}. \tag{3.27}$$

For configurations with one representation ($L=1$), one finds

$$\begin{aligned}
F^c_{(1,0,\ldots)} &= i^{1-c} f^c_{\ydiagram{1}}, \qquad F^c_{(2,0,\ldots)} = \frac{i^{-c}}{2}\left(f^c_{\ydiagram{2}} + f^c_{\ydiagram{1,1}}\right), \qquad F^c_{(0,1,0,\ldots)} = \frac{i^{1-c}}{2}\left(f^c_{\ydiagram{2}} - f^c_{\ydiagram{1,1}}\right)\\
F^c_{(3,0,\ldots)} &= -\frac{i^{1-c}}{6}\left(f^c_{\ydiagram{3}} + 2f^c_{\ydiagram{2,1}} + f^c_{\ydiagram{1,1,1}}\right), \qquad F^c_{(1,1,0,\ldots)} = \frac{i^{-c}}{2}\left(f^c_{\ydiagram{3}} - f^c_{\ydiagram{1,1,1}}\right)\\
F^c_{(0,0,1,0,\ldots)} &= \frac{i^{1-c}}{3}\left(f^c_{\ydiagram{3}} - f^c_{\ydiagram{2,1}} + f^c_{\ydiagram{1,1,1}} + f^c_{\ydiagram{1}}(q^3,\mathrm{e}^{-3t})\right),
\end{aligned} \tag{3.28}$$

For configurations with two representations ($L=2$), one finds

$$\begin{aligned}
F^c_{((1,0,\ldots),(1,0,\ldots))} &= i^{-c} f^c_{\ydiagram{1}\,\ydiagram{1}}, \qquad F^c_{((2,0,\ldots),(1,0,\ldots))} = -\frac{i^{1-c}}{2}\left(f^c_{\ydiagram{2}\,\ydiagram{1}} + f^c_{\ydiagram{1,1}\,\ydiagram{1}}\right),\\
F^c_{((0,1,0,\ldots),(1,0,\ldots))} &= \frac{i^{-c}}{2}\left(f^c_{\ydiagram{2}\,\ydiagram{1}} - f^c_{\ydiagram{1,1}\,\ydiagram{1}}\right).
\end{aligned} \tag{3.29}$$

Using the above formulae, we can compute the $F^c_{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}$ generating functionals and put them in the form of (3.27) by expanding in g_s. This will extract the open Gromov-Witten invariants from our previous results.

JHEP02(2005)002

I. The SO/Sp framed unknot. The topological vertex gives the following results:

$$
\begin{aligned}
F^{c=1}_{(1,0,0,\dots)} &= (-1)^p Q^{1/2}\,,\\
F^{c=1}_{(2,0,0,\dots)} &= \frac{1}{2}\left[(1+p)^2 Q^{1/2}(1-Q)\right] g_s - \frac{1}{48}\left[(1+p)^2(1+4p+2p^2)Q^{1/2}(1-Q)\right] g_s^3 + \cdots\,,\\
F^{c=1}_{(0,1,0,\dots)} &= \left[(1+p)Q^{1/2}(1-Q)\right] - \frac{1}{24}\left[(3+11p+12p^2+4p^3)Q^{1/2}(1-Q)\right] g_s^2 + \cdots\,,\\
F^{c=1}_{(3,0,0,\dots)} &= \frac{1}{6}\left[(-1)^p Q^{1/2}(1+p)^3(1+3p-6Q(1+p)+Q^2(5+3p))\right] g_s^2 + \cdots\,,\\
F^{c=1}_{(1,1,0,\dots)} &= \left[(-1)^p Q^{1/2}(1+p)^2(1+2p-4Q(1+p)+Q^2(3+2p))\right] g_s + \cdots\,,\\
F^{c=1}_{(0,0,1,\dots)} &= \frac{1}{6}\left[(-1)^p Q^{1/2}(3(1+p)(2+3p+Q^2(4+3p))-2Q(8+18p+9p^2))\right] + \cdots\,.
\end{aligned}
$$

In order to compare with the localization computation, we introduce first some notation. We will consider the following real torus action on the resolved conifold X:

$$
e^{i\phi}\cdot(X_1,X_2,X_3,X_4) = (e^{i\lambda_1\phi}X_1, e^{i\lambda_2\phi}X_2, e^{i\lambda_3\phi}X_3, e^{i\lambda_4\phi}X_4)\,.
$$

The weights of the torus action on the local coordinates $z = X_1/X_2, u = X_2X_3, v = X_2X_4$ are given by $\lambda_z = \lambda_1 - \lambda_2, \lambda_u = \lambda_2 + \lambda_3, \lambda_v = \lambda_2 + \lambda_4$ respectively. Note that from the compatibility of the torus action with the antiholomorphic involution it follows that $\lambda_u + \lambda_v + \lambda_z = 0$. Now we can present the localization results:

$$
\begin{aligned}
F^{c=1}_{(1,0,0,\dots)} &= Q^{\frac{1}{2}}\,,\\
F^{c=1}_{(2,0,0,\dots)} &= \frac{1}{2}\left[\left(\frac{a}{a-1}\right)^2 Q^{\frac{1}{2}}(1-Q)\right] g_s - \frac{1}{48}\left[\frac{a^2(a^2+2a-1)}{(a-1)^4}Q^{\frac{1}{2}}(1-Q)\right] g_s^3 + \cdots\,,\\
F^{c=1}_{(0,1,0,\dots)} &= \left[\frac{a}{a-1}Q^{\frac{1}{2}}(1-Q)\right] - \frac{1}{24}\left[\frac{a(a+1)(3a-1)}{(a-1)^3}Q^{\frac{1}{2}}(1-Q)\right] g_s^2 + \cdots\,,\\
F^{c=1}_{(3,0,0,\dots)} &= -\frac{1}{6}\left[Q^{1/2}\left(\frac{a}{a-1}\right)^3\left(\frac{a+2}{a-1} - \frac{6a}{a-1}Q + \dots\right)\right] g_s^2 + \cdots\,,\\
F^{c=1}_{(1,1,0,\dots)} &= -\left[Q^{1/2}\left(\frac{a}{a-1}\right)^2\left(\frac{a+1}{a-1} - \frac{4a}{a-1}Q + \dots\right)\right] g_s + \cdots\,,\\
F^{c=1}_{(0,0,1,\dots)} &= -\frac{1}{6}\left[Q^{1/2}\left(\frac{3a(2a+1)}{(a-1)^2} - \frac{2(8a^2+2a-1)}{(a-1)^2}Q + \dots\right)\right] + \dots\,,
\end{aligned}
$$

where $a = -\lambda_v/\lambda_z$. After making the substitution $a = 1 + \frac{1}{p}$, we find that the above results coincide with the expressions obtained from the vertex computation up to factors of $\pm(-1)^p$. The sign difference is due to different choice of conventions between the vertex and the localization computations. As an example, we present below the graphs contributing to the unoriented open Gromov-Witten invariant for genus 1 maps with degree 3 $\mathbb{RP}^2$ and winding vector $(0,1,0,\dots)$, as well as their contributions. The contributions of the above graphs are computed according to the rules explained in [9]. We obtain:

$$
C^{(1,3),(a)}_{(0,1,0,\dots)} = \frac{(\lambda_u - 2\lambda_v)(2\lambda_u - \lambda_v)\lambda_v(\lambda_u + 2\lambda_v)(\lambda_u^3 + 6\lambda_u^2\lambda_v + \lambda_u\lambda_v^2 + 2\lambda_v^3)}{48\lambda_u^3\lambda_z^4}\,,
$$

JHEP02(2005)002

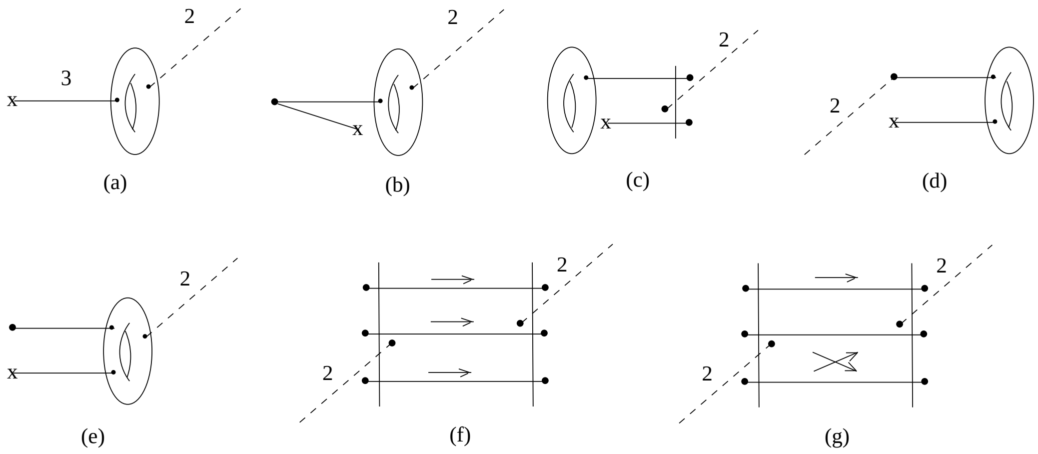


Figure 6: One crosscap, genus 1 and three crosscaps, genus 0 at degree 3 $\mathbb{RP}^2$, and winding vector $(0,1,0,\ldots)$.

$$\begin{aligned}
C^{(1,3),(b)}_{(0,1,0,\ldots)} &= -\frac{(\lambda_u-2\lambda_v)\lambda_v^2(\lambda_u+2\lambda_v)}{48\lambda_u^2\lambda_z^2}\,, \quad & C^{(1,3),(c)}_{(0,1,0,\ldots)} &= \frac{\lambda_v^2(\lambda_u+2\lambda_v)}{24\lambda_u\lambda_z^2}\,,\\
C^{(1,3),(d)}_{(0,1,0,\ldots)} &= -\frac{\lambda_v^2(2\lambda_u+\lambda_v)(\lambda_u+2\lambda_v)}{24\lambda_z^4}\,, \quad & C^{(1,3),(e)}_{(0,1,0,\ldots)} &= \frac{\lambda_v^3(\lambda_u+2\lambda_v)(\lambda_u^3-2\lambda_u\lambda_v^2-2\lambda_v^3)}{12\lambda_u^3\lambda_z^4}\,,\\
C^{(1,3),(f)}_{(0,1,0,\ldots)} &= -\frac{(\lambda_u+2\lambda_v)\lambda_v^3(\lambda_u-2\lambda_v)}{6\lambda_u\lambda_z^4}\,, \quad & C^{(1,3),(g)}_{(0,1,0,\ldots)} &= \frac{(\lambda_u+2\lambda_v)\lambda_v^3(\lambda_u-2\lambda_v)}{2\lambda_u\lambda_z^4}\,.
\end{aligned}$$

II. $\mathbb{P}^2$ attached to $\mathbb{RP}^2$. The topological vertex gives the following results:

$$\begin{aligned}
F^{c=1}_{(1,0,\ldots)} &= (-1)^p Q^{1/2} e^{-t}[-1-2(-2+Q)e^{-t}+(-35+30Q-3Q^2)e^{-2t}+\\
&\qquad\qquad + 4(100-122Q+33Q^2)e^{-3t}+\cdots]+\cdots\,,\\
F^{c=2}_{(1,0,\ldots)} &= -(-1)^p Q^2 e^{-3t}\left[1+(-15+2Q^2)e^{-t}+\ldots\right]g_s+\cdots\,,\\
F^{c=1}_{(2,0,\ldots)} &= \frac{1}{2}Q^{1/2}e^{-t}\left[-p^2+(3+6p+6p^2)e^{-t}+\cdots\right]g_s+\cdots\,,\\
F^{c=2}_{(2,0,\ldots)} &= -\frac{1}{2}Q^2 e^{-3t}\left[1+2p+2p^2+\ldots\right]g_s^2+\cdots\,,\\
F^{c=1}_{(0,1,0,\ldots)} &= Q^{1/2}e^{-t}\left[-p+(3+6p)e^{-t}+\cdots\right]+\cdots\,,\\
F^{c=2}_{(0,1,0,\ldots)} &= -Q^2e^{-3t}[1+2p+\cdots]g_s+\cdots\,.
\end{aligned}$$

For the localization computations, we will use the same notation as in [5]. We present below some of the localization computations we performed. First, we obtain

$$F^{c=1}_{(2,0,\ldots)} = \frac{1}{2}Q^{1/2}e^{-t}\left[-\left(\frac{\lambda_v-\lambda_u}{\lambda_v-2\lambda_u}\right)^2+\left(\frac{3(2\lambda_u^2-2\lambda_u\lambda_v+\lambda_v^2)}{(2\lambda_u-\lambda_v)^2}\right)e^{-t}+\cdots\right]g_s+\cdots\,.$$

We present the graphs contributing at degree 2 hyperplane class in the expression above, as well as their contributions.

$$C^{(2,0,0,\ldots),(a)}_{(-1,1,2)} = \frac{\lambda_v(\lambda_u-\lambda_v)^2(3\lambda_u-2\lambda_v)}{\lambda_u^2(\lambda_v-2\lambda_u)^2}\,, \quad C^{(2,0,0,\ldots),(b)}_{(-1,1,2)} = \frac{(\lambda_u-\lambda_v)^4}{\lambda_u^2(\lambda_v-2\lambda_u)^2}\,,$$

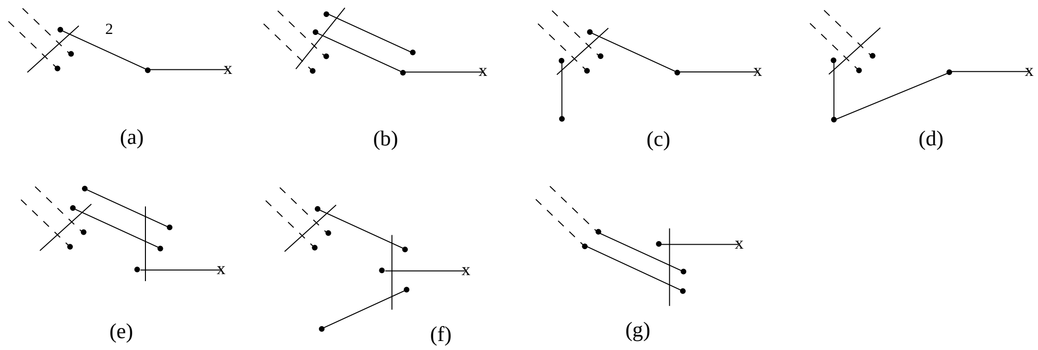


Figure 7: One crosscap graphs at degree 1 $\mathbb{RP}^2$, degree 2 hyperplane and winding vector $(2, 0, \ldots)$.

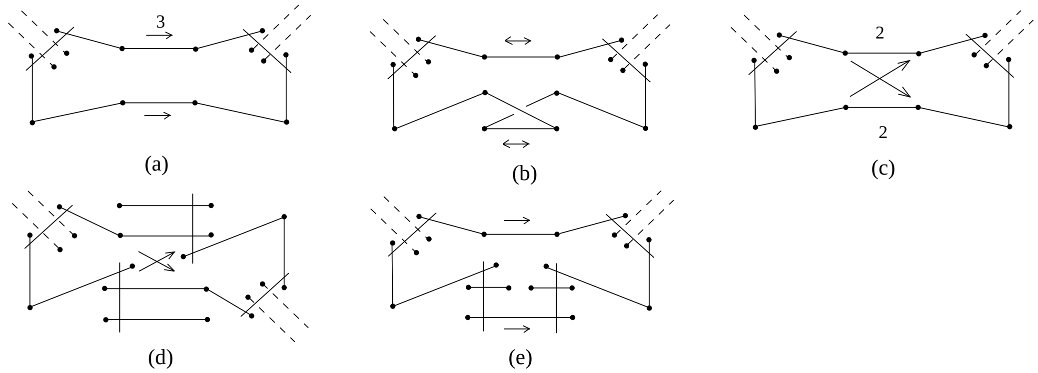


Figure 8: Two crosscaps graphs at degree 4 $\mathbb{RP}^2$, degree 3 hyperplane and winding vector $(2, 0, \ldots)$.

$$C^{(2,0,0,\ldots),(c)}_{(-1,1,2)} = \frac{2\lambda_u^2 - 2\lambda_u\lambda_v + \lambda_v^2}{2(\lambda_v - 2\lambda_u)^2}, \quad C^{(2,0,0,\ldots),(d)}_{(-1,1,2)} = \frac{\lambda_u^2}{2(\lambda_v - 2\lambda_u)^2},$$

$$C^{(2,0,0,\ldots),(e)}_{(-1,1,2)} = \frac{\lambda_v^2(\lambda_u - \lambda_v)^2}{2\lambda_u^2(\lambda_v - 2\lambda_u)^2}, \quad C^{(2,0,0,\ldots),(f)}_{(-1,1,2)} = \frac{(\lambda_u - \lambda_v)^2}{2(\lambda_v - 2\lambda_u)^2}, \quad C^{(2,0,0,\ldots),(g)}_{(-1,1,2)} = \frac{\lambda_v^2}{2\lambda_u^2}.$$

In the expressions above, the subscript of the contributions is (χ, d_1, d_2) where χ is the unoriented genus of the closed component of the map and d_1 and d_2 are the $\mathbb{RP}^2$ and hyperplane degrees respectively. Then, for the same winding vector, at 2 crosscaps we obtain

$$F^{c=2}_{(2,0,\ldots)} = -\frac{1}{2}Q^2 e^{-3t}\left[\frac{2\lambda_u^2 - 2\lambda_u\lambda_v + \lambda_v^2}{(\lambda_v - 2\lambda_u)^2} + \cdots\right] + \cdots.$$

The two crosscaps configurations were discussed at length in [5]. The graphs come in sets and there is a single set such that the sum of the contributions of the corresponding graphs does not vanish. That set and the graphs contributions are presented below.

$$C^{(2,0,0,\ldots),(a)}_{(0,4,3)} = -\frac{(\lambda_u^2 - \lambda_u\lambda_v + \lambda_v^2)(2\lambda_u^2 - 2\lambda_u\lambda_v + \lambda_v^2)}{2\lambda_z^2(\lambda_v - 2\lambda_u)^2},$$

$$C^{(2,0,0,\ldots),(b)}_{(0,4,3)} = \frac{1}{2}C^{(2,0,0,\ldots),(c)}_{(0,4,3)} = -\frac{\lambda_u\lambda_v(2\lambda_u^2 - 2\lambda_u\lambda_v + \lambda_v^2)}{2\lambda_z^2(\lambda_v - 2\lambda_u)^2},$$

$$C^{(2,0,0,\ldots),(d)}_{(0,4,3)} = -C^{(2,0,0,\ldots),(e)}_{(0,4,3)} = \frac{(\lambda_u^2 + \lambda_v^2)(2\lambda_u^2 - 2\lambda_u\lambda_v + \lambda_v^2)}{2\lambda_z^2(\lambda_v - 2\lambda_u)^2}.$$

We also obtain

$$
\begin{aligned}
F^{c=1}_{(0,1,0,\ldots)} &= Q^{1/2}e^{-t}\left[\frac{\lambda_v-\lambda_u}{\lambda_v-2\lambda_u} - 3\left(\frac{\lambda_v}{\lambda_v-2\lambda_u}\right)e^{-t}+\cdots\right]+\cdots\,,\\
F^{c=2}_{(0,1,0,\cdots)} &= Q^{2}e^{-3t}\left[\left(\frac{\lambda_v}{\lambda_v-2\lambda_u}\right)+\cdots\right]+\cdots\,.
\end{aligned}
\tag{3.30}
$$

We note that for this geometry we obtain agreement with the vertex computation if we set $p=-\frac{\lambda_v-\lambda_u}{\lambda_v-2\lambda_u}$.

III. SO / Sp **Hopf link invariant.** The results obtained from the topological vertex are:

$$
\begin{aligned}
F^{c=1}_{((1,0,\ldots),(1,0,\ldots))} &= (-1)^{p_1+p_2}Q^{1/2}(1-Q)g_s - \frac{1}{24}(-1)^{p_1+p_2}Q^{1/2}(1-Q)g_s^3+\cdots\,,\\
F^{c=1}_{((2,0,\ldots),(1,0,\ldots))} &= \frac{1}{2}(-1)^{p_2}Q^{1/2}\left[1+2p_1+2p_1^2-2Q(1+2p_1^2)+Q^2(1-2p_1+2p_1^2)\right]g_s^2+\cdots\,,\\
F^{c=1}_{((0,1,0,\ldots),(1,0,\ldots))} &= (-1)^{p_2}Q^{1/2}\left[1+2p_1-4Qp_1-Q^2(1-2p_1)\right]g_s+\cdots\,.
\end{aligned}
\tag{3.31}
$$

The localization results are:

$$
\begin{aligned}
F^{c=1}_{((1,0,\ldots),(1,0,\ldots))} &= Q^{1/2}(1-Q)g_s - \frac{1}{24}Q^{1/2}(1-Q)g_s^3+\cdots\,,\\
F^{c=1}_{((2,0,\ldots),(1,0,\ldots))} &= -\frac{1}{2}Q^{1/2}\left[\frac{\lambda_u^2+2\lambda_u\lambda_v+2\lambda_v^2}{\lambda_u^2} - 2Q\left(\frac{{\lambda_u}^2+2{\lambda_v}^2}{{\lambda_u}^2}\right)+\right.\\
&\qquad\left.+\,Q^2\left(\frac{\lambda_u^2-2\lambda_u\lambda_v+2\lambda_v^2}{\lambda_u^2}\right)\right]g_s^2+\cdots\,,\\
F^{c=1}_{((0,1,0,\ldots),(1,0,\ldots))} &= Q^{1/2}\left[\frac{\lambda_u+2\lambda_v}{\lambda_u} - 4Q\left(\frac{\lambda_v}{\lambda_u}\right) - Q^2\left(\frac{\lambda_u-2\lambda_v}{\lambda_u}\right)\right]g_s+\cdots\,.
\end{aligned}
\tag{3.32}
$$

To obtain agreement with the vertex result for this D-brane configuration, we need to set $p_1=\lambda_v/\lambda_u$. These computations offer strong evidence of the equivalence between the vertex computation and the localization on the moduli space of stable open unoriented maps.

4. Application: the BPS structure of the coloured Kauffman polynomial

One of the most interesting applications of the above results is the determination of the BPS structure of the coloured Kauffman polynomial. In contrast to the results obtained for orientifolds of toric geometries above, we won't be able to give a full determination of all quantities involved for arbitrary knots, but we can still formulate some interesting structural properties of the knot polynomials similar to those explained in [38, 23, 25, 24]. We will first recall the results for the coloured HOMFLY polynomial, and then we will state and illustrate the results for the coloured Kauffman polynomial.

4.1 Chern-Simons invariants and knot polynomials

Let us consider Chern-Simons theory on $\mathbf{S}^3$ with gauge group G. The natural operators in this theory are the holonomies of the gauge connection around a knot $\mathcal{K}$,

$$
W^{\mathcal{K}}_R(A) = \mathrm{P}\ \exp\oint_{\mathcal{K}} A\,. \tag{4.1}
$$

If we now consider a link $\mathcal{L}$ with components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$, the correlation function

$$W^G_{R_1 \cdots R_L}(\mathcal{L}) = \langle W^{\mathcal{K}_1}_{R_1} \cdots W^{\mathcal{K}_L}_{R_L} \rangle \tag{4.2}$$

defines a topological invariant of the link $\mathcal{L}$. In this equation the bracket denotes a normalized vacuum expectation value, and we have indicated the gauge group G as a superscript. It is well-known [42] that Chern-Simons produces in fact invariants of *framed* links, but in the following we will consider knots in the so-called standard framing (see [15, 31] for a review of these topics). The correlation functions (4.2) turn out to be rational functions of the variables $q^{\pm 1/2}$, $\lambda^{\pm 1/2}$. The variable q is defined as $q = e^{i g_s}$, where g_s is the effective Chern-Simons coupling constant

$$g_s = \frac{2\pi}{k+y}, \tag{4.3}$$

k is the coupling constant of Chern-Simons theory, and y is the dual Coxeter of the gauge group (therefore it is N for $\mathrm{U}(N)$, $N-2$ for $\mathrm{SO}(N)$, and $N+1$ for $\mathrm{Sp}(N)$). The variable λ is defined by

$$\lambda = q^{N+a}, \tag{4.4}$$

where

$$a = \begin{cases} 0 & \text{for} \quad \mathrm{U}(N)\,, \\ -1 & \text{for} \quad \mathrm{SO}(N)\,, \\ 1 & \text{for} \quad \mathrm{Sp}(N)\,. \end{cases} \tag{4.5}$$

The vacuum expectation values of Wilson loops are related to link invariants obtained from quantum groups [42]:

1. If $G = \mathrm{U}(N)$ and $R_1 = \cdots = R_L = \square$, then

$$W^{\mathrm{U}(N)}_{\square \cdots \square}(\mathcal{L}) = \lambda^{\mathrm{lk}(\mathcal{L})} \left(\frac{\lambda^{1/2} - \lambda^{-1/2}}{q^{1/2} - q^{-1/2}} \right) P_{\mathcal{L}}(q, \lambda)\,, \tag{4.6}$$

   where $P_{\mathcal{L}}(q,\lambda)$ is the HOMFLY polynomial of $\mathcal{L}$ [11], and $\mathrm{lk}(\mathcal{L})$ is its linking number.

2. If $G = \mathrm{SO}(N)$ and $R_1 = \cdots = R_L = \square$, then

$$W^{\mathrm{SO}(N)}_{\square \cdots \square}(\mathcal{L}) = \lambda^{\mathrm{lk}(\mathcal{L})} \left(1 + \frac{\lambda^{1/2} - \lambda^{-1/2}}{q^{1/2} - q^{-1/2}} \right) F_{\mathcal{L}}(q, \lambda)\,, \tag{4.7}$$

   where $F_{\mathcal{L}}(q,\lambda)$ is the Kauffman polynomial of $\mathcal{L}$ [20].

We will call $W^{\mathrm{U}(N)}_{R_1 \ldots R_L}(\mathcal{L})$ and $W^{\mathrm{SO}(N)}_{R_1 \ldots R_L}(\mathcal{L})$ the coloured HOMFLY and Kauffman polynomials of $\mathcal{L}$, respectively. Note that there is a slight abuse of language here, since these Chern-Simons correlation functions are not polynomials, but rather rational functions.

4.2 BPS structure: statement and examples

In [38], Ooguri and Vafa extended the duality of [13] between Chern-Simons theory on $\mathbf{S}^3$ and topological strings on the resolved conifold by incorporating the correlation functions (4.2). We will consider the case of knots, although everything we will say has a

JHEP02(2005)002

straigthforward generalization to links. The results of [38] are the following: first, to any knot $\mathcal{K} \in \mathbf{S}^3$ one can associate a lagrangian submanifold $S_{\mathcal{K}}$ in the resolved conifold. Moreover, the generating functional of knot invariants

$$Z_{\mathrm{U}(N)}(V) = \sum_R W_R^{\mathrm{U}(N)}(\mathcal{K}) \mathrm{Tr}_R V \,, \tag{4.8}$$

where V is a $\mathrm{U}(M)$ matrix, is the partition function for open topological strings propagating on the resolved conifold and with Dirichlet boundary conditions associated to $S_{\mathcal{K}}$ (after some appropriate analytic continuation). Equivalently, we consider M branes wrapping $S_{\mathcal{K}}$, where M is the rank of V, and compute the partition function of topological string theory in this D-brane background. Since open string amplitudes have the BPS structure explained in (2.5) and (2.6), this leads to structure results for the knot invariants $W_R^{\mathrm{U}(N)}(\mathcal{K})$ (which play the rôle of Z_R). This is explained in detail in [23, 25, 24].

The large N duality of [13] can be generalized by considering an orientifold of the two geometries involved in the geometric transition, namely the resolved and the deformed conifold [40]. The deformed conifold is defined by the equation $z_1 z_4 - z_2 z_3 = \mu$ and it contains an $\mathbf{S}^3$. If we wrap $2N$ branes on the three-sphere, the spacetime description of the open topological string theory is Chern-Simons theory on $\mathbf{S}^3$ with gauge group $\mathrm{U}(2N)$ and at level k (the level is related to the open string coupling constant). We now consider the following involution of the geometry

$$I : (z_1, z_2, z_3, z_4) \to (\bar{z}_4, -\bar{z}_3, -\bar{z}_2, \bar{z}_1) \tag{4.9}$$

that leaves the $\mathbf{S}^3$ invariant. The string field theory for the resulting open strings is now Chern-Simons theory with gauge group $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$, depending on the choice of orientifold action on the gauge group. The orientifold action on the resolved conifold is given by (3.5). It then follows from the results of [38] and the orientifold action considered in [40] that the Chern-Simons generating functional

$$Z_{\mathrm{SO}\,/\,\mathrm{Sp}}(V) = \sum_R W_R^{\mathrm{SO}\,/\,\mathrm{Sp}}(\mathcal{K}) \mathrm{Tr}_R V \,, \tag{4.10}$$

where V is again a $\mathrm{U}(M)$ matrix, is the total partition function for open strings propagating on the orientifold of the resolved conifold with M branes wrapping $S_{\mathcal{K}}$. In particular, the logarithm of (4.10) will have the structure explained in (2.15), where the oriented contribution is obtained by considering a covering geometry with both $S_{\mathcal{K}}$ and its image under the involution (3.5), $I(S_{\mathcal{K}})$. We can then translate the structure results presented in section 2 into structure results for the coloured Kauffman polynomial.

The main problem in making this translation precise is that, given an arbitrary knot $\mathcal{K}$, we lack a precise prescription to compute the contribution of the covering amplitude. The covering amplitude $\mathcal{C}_{R_1 R_2}$ is defined as the oriented amplitude in the covering geometry in the presence of two sets of branes wrapping $S_{\mathcal{K}}$ and $I(S_{\mathcal{K}})$, with representations R_1, R_2, respectively. If one of the representations is trivial, we recover the oriented amplitude in the presence of $S_{\mathcal{K}}$, therefore $\mathcal{C}_R = W_R^{\mathrm{U}(N)}(\mathcal{K})$. But in the general case it is not obvious how to determine $\mathcal{C}_{R_1 R_2}$. Although there are proposals for the geometry of the lagrangian submanifolds $S_{\mathcal{K}}$ [25, 41], a direct Gromov-Witten computation of the corresponding open string amplitudes seems to be very difficult. One possible way of determining $\mathcal{C}_{R_1R_2}$ would be to translate it into a pure knot-theoretic computation in the context of Chern-Simons theory, but we haven't found a completely satisfactory solution to this problem.

Although we don't know how to compute the covering amplitude for an arbitrary knot, we can still extract the $\widehat{f}_R^{c=1}$ amplitudes from the knowledge of $W_R^{\mathrm{SO}(N)}(\mathcal{K})$. This goes as follows. Let us define the rational functions $g_R(q,\lambda)$ through the following equation

$$\log Z_{\mathrm{SO}}(V) = \sum_R \sum_{d\,\mathrm{odd}} \frac{1}{d} g_R(q^d,\lambda^d)\mathrm{Tr}_R V^d\,, \tag{4.11}$$

and define as well

$$\widehat{g}_R(q,\lambda) = \sum_{RR'} M^{-1}_{RR'}(q) g_R(q,\lambda)\,. \tag{4.12}$$

Clearly, since we are not substracting the covering piece in the l.h.s. of (4.11), we cannot expect much structure for $\widehat{g}_R(q,\lambda)$. However, one has that

$$\widehat{f}_R^{c=1}(q,\lambda) = \frac{1}{2}\left(\widehat{g}_R(q,\lambda^{1/2}) - (-1)^{\ell(R)}\widehat{g}_R(q,-\lambda^{1/2})\right). \tag{4.13}$$

This follows from parity considerations. The invariants $W_R^{\mathrm{U}(N)}(\mathcal{K})$ have powers of $\lambda^{\frac{1}{2}}$ of the form $\ell(R)+2k$, while $W_R^{\mathrm{SO}(N)}(\mathcal{K})$ have powers of $\lambda^{\frac{1}{2}}$ both of the form $\ell(R)+2k$ and $\ell(R)+2k+1$. The first ones correspond to both oriented and $c=2$ contributions, while the last ones correspond to $c=1$ contributions. Also, the covering contribution $\mathcal{C}_{R_1R_2}$ (being an oriented amplitude) contains only powers in $\lambda^{\frac{1}{2}}$ of the form $\ell(R_1)+\ell(R_2)+2k$. It is now easy to see from the results in section 2 that $\widehat{f}_R^{c=1}$ does not involve at all the covering contributions, and can be computed solely from the $\mathrm{SO}(N)$ invariants, precisely in the way specified by (4.13). We can then formulate the following conjecture concerning the structure of the coloured Kauffman polynomial:

Conjecture. *Let $\widehat{g}_R(q,\lambda)$ be defined in terms of the coloured Kauffman polynomial by (4.11) and (4.12). Then, we have that*

$$\frac{1}{2}\left(\widehat{g}_R(q,\lambda^{1/2}) - (-1)^{\ell(R)}\widehat{g}_R(q,-\lambda^{1/2})\right) = \sum_{g,\beta} N^{c=1}_{R,g,\beta}\left(q^{1/2}-q^{-1/2}\right)^{2g}\lambda^\beta\,, \tag{4.14}$$

where $N^{c=1}_{R,g,\beta}$ are integer *numbers. They are BPS invariants corresponding to unoriented open string amplitudes with one crosscap.*

In the case of $W_\Box^{\mathrm{SO}(N)}(q,\lambda)$, which is the unnormalized Kauffman polynomial, we can be slightly more precise, since we know that $\mathcal{C}_\Box(q,\lambda) = W_\Box^{\mathrm{U}(N)}(q,\lambda)$, which is the unnormalized HOMFLY polynomial. We then deduce that

$$\begin{aligned} &W_\Box^{\mathrm{SO}(N)}(q,\lambda) - W_\Box^{\mathrm{U}(N)}(q,\lambda) = \\ &= \sum_{g,\beta} N^{c=1}_{\Box,g,\beta}(q^{1/2}-q^{-1/2})^{2g}\lambda^\beta + \sum_{g,\beta} N^{c=2}_{\Box,g,\beta}(q^{1/2}-q^{-1/2})^{2g+1}\lambda^\beta\,. \end{aligned} \tag{4.15}$$

On the other hand, it follows from integrality of the oriented amplitudes that

$$W_{\square}^{\mathrm{U}(N)}(q,\lambda) = \sum_{g\geq 0} p_g^H(\lambda)(q^{1/2}-q^{-1/2})^{2g-1}\,, \tag{4.16}$$

where $p_g^H(\lambda)$ is an odd polynomial in $\lambda^{\pm 1/2}$. It then follows that the structure of the unnormalized Kauffman polynomial is given by

$$W_{\square}^{\mathrm{SO}(N)}(q,\lambda) = \sum_{b\geq 0} p_b^K(\lambda)(q^{1/2}-q^{-1/2})^{b-1}\,, \tag{4.17}$$

where $p_b^K(\lambda)$ is an odd (even) polynomial in $\lambda^{\pm 1/2}$ for b even (odd). Moreover,

$$p_0^K(\lambda) = p_0^H(\lambda)\,. \tag{4.18}$$

This structural prediction turns out to be a well-known result in the theory of the Kauffman polynomial, see for example [28, page 183]. One can easily compute $N_{\square,g,\beta}^{c=1,2}$ for various knots by computing the corresponding Kauffman polynomial. For example, the results of [26] imply that

$$N_{\square,g,\beta}^{c=2} = 0 \tag{4.19}$$

for all torus knots.

Let us now turn to checks of the conjecture above for different knots and higher representations. The simplest case is of course the unknot, but this case has been already checked in section 3 (indeed, in the case of the unknot we know even how to compute the covering amplitude for arbitrary representations). In order to test the conjecture, we have to compute the invariants $W_R^{\mathrm{SO}(N)}(\mathcal{K})$ for arbitrary R. A class of nontrivial knots where this is doable are torus knots. In the case of $\mathrm{U}(N)$ invariants, this was done in [23] by using the formalism of knot operators [21] and the results of [22]. For $\mathrm{SO}(N)$, the formalism of knot operators was used in [26] to compute invariants in the fundamental representation, but this has not been generalized to higher representations. For torus knots of the form $(2,m)$, however, one can use the results of [39] to write down a formula for the invariants in any representation of any gauge group. The formula reads as follows:

$$\mathcal{W}_R^G(\mathcal{K}_{(2,m)}) = \sum_{S\in R\otimes R} (\dim_q S)(c_S(R,R))^m\,, \tag{4.20}$$

where

$$c_S(R_1,R_2) = \epsilon_{R_1R_2}^S q^{\frac{C_{R_1}+C_{R_2}}{2}-\frac{C_S}{4}}\,. \tag{4.21}$$

In this equation, C_R is the quadratic Casimir

$$C_R = \kappa_R + \ell(R)(N+a)\,, \tag{4.22}$$

where a is given in (4.5), and $\epsilon_{R_1R_2}^S$ is a sign which counts whether S appears symmetrically or antisimmetrically in the tensor product $R_1\otimes R_2$. In case S appears with no multiplicity, there is an explicit expression for this sign given by [35]

$$\epsilon_{R_1R_2}^S = (-1)^{\rho\cdot(\Lambda_1+\Lambda_2-\Lambda_S)}\,, \tag{4.23}$$

	$\beta=1$	2	3
$g=0$	3	-3	1
1	1	-1	0

Table 1: BPS invariants $N^{c=1}_{\square,g,\beta}$ for the trefoil knot.

	$\beta=3/2$	$5/2$	$7/2$	$9/2$	$11/2$
$g=0$	8	-39	69	-53	15
1	6	-61	146	-126	35
2	1	-37	128	-120	28
3	0	-10	56	-55	9
4	0	-1	12	-12	1
5	0	0	1	-1	0

Table 2: BPS invariants $N^{c=1}_{\square\square,g,\beta}$ for the trefoil knot.

	$\beta=3/2$	$5/2$	$7/2$	$9/2$	$11/2$
$g=0$	16	-69	111	-79	21
1	20	-146	307	-251	70
2	8	-128	366	-330	84
3	1	-56	230	-220	45
4	0	-12	79	-78	11
5	0	-1	14	-14	1
6	0	0	1	-1	0

Table 3: BPS invariants $N^{c=1}_{\substack{\square\\\square},g,\beta}$ for the trefoil knot.

where Λ_1, Λ_2, Λ_S are the highest weights of to the representations R_1, R_2, S, respectively. Using (4.20) one can easily compute the invariants of the $(2,m)$ torus knots in the $\mathrm{SO}(N)$ case, and extract g_R (hence $N^{c=1}_{R,g,\beta}$) for various representations. In all cases we have found agreement with the above conjecture. We now present some results for the BPS invariants for the simplest torus knot, the $(2,3)$ knot or trefoil knot, for representations up to three boxes. All the invariants that are not shown in the Tables are understood to be zero.

The results for representations with three boxes are listed in appendix C.

Although we have focused in this section on the case of knots, it is straightforward to extend the conjecture above to the case of links, and extract the $c=1$ piece from the SO Chern-Simons invariants. Framed knots can be also considered by using exactly the same rules that are used for $\mathrm{U}(N)$ invariants [32].

5. Discussion

In this paper we extended the results of [5] in order to study open string amplitudes on orientifolds without fixed planes. We found the general structure of the twisted and untwisted contributions, we determined the BPS structure of the corresponding amplitudes, and we checked our results in various examples. We want to remark that, although our main testing ground have been orientifolds of noncompact, toric Calabi-Yau orientifolds with noncompact branes, the general results about the structure and integrality properties of the amplitudes should be valid in general.

One of the motivations of the present paper was to extend the results of [25, 24] on the BPS structure of the coloured HOMFLY polynomial to the coloured Kauffman polynomial. Although our general structural results on open string amplitudes on orientifolds give a first principles answer to this problem, as it has been made clear in the analysis of the framed unknot and the Hopf link, we haven't been able to determine the covering contribution for arbitrary knots. This is an important open issue that one should resolve in order to obtain a complete picture of the correspondence between enumerative geometry and knot invariants implied by large N dualities.

We have also seen that the predictions obtained from the topological vertex in the unoriented case agree with unoriented localization computations in the examples that we have worked out. It would be very interesting to derive a more general and precise correspondence between these two approaches, following the lines of the mathematical treatment of the vertex given in [6, 27], and maybe connect the unoriented Gromov-Witten theory sketched here and in [9, 5], with a moduli problem involving ideal sheaves, generalizing in this way the results of [33]. Finally, our results both in this paper and in [5] cover only orientifolds without fixed planes, and more work is needed in order to understand the general situation from the point of view of topological string theory.

Acknowledgments

M.M. would like to thank Cumrun Vafa and specially José Labastida for discussions on SO(N) Chern-Simons invariants and topological strings. The work of V.B. was supported by a Rhodes Scholarship and a NSERC PGS Doctoral Scholarship. The work of B.F. was supported in part by DOE grant DE-FG02-96ER40959.

A. Useful formulae and Schur functions.

In this appendix we will list some useful identities of Schur functions and their relations to the unknot and Hopf link invariants. For a more detailed discussion of Schur functions see for example [30, 12]. Applications of these results to topological string computations can be found in [37, 10, 16, 18].

Let R be a partition associated to a Young tableau. Let $\ell(R)$ be the number of boxes of the Young tableau and $l_i(R)$ be the number of boxes in the i-th row. We define the quantity

$$W_R(q) = s_R(q^\rho)\,, \tag{A.1}$$

where $s_R(q^\rho)$ is the Schur function with the substitution $s_R(x_i = q^{-i+1/2})$, where i runs from 1 to ∞. $W_R(q)$ is the leading order of the U(N) quantum dimension $\dim_q^{\mathrm{U}(N)} R$ (in the sense defined in [3]). We also recall the general formula for quantum dimensions of a group G:

$$\dim_q^G R = \prod_{\alpha\in\Delta_+} \frac{[(\Lambda_R+\rho,\alpha)]}{[(\rho,\alpha)]}\,, \tag{A.2}$$

where Λ_R is the highest weight of the representation R, ρ is the Weyl vector, and the product is over the positive roots of G. We also defined the following q-number:

$$[x] = q^{x/2} - q^{-x/2}\,. \tag{A.3}$$

Another important object is

$$W_{R_1R_2}(q) = s_{R_1}(q^\rho) s_{R_2}(q^{\ell(R_1)+\rho})\,, \tag{A.4}$$

where $s_{R_2}(q^{\ell(R_2)+\rho}) = s_{R_2}(x_i = q^{l_i(R_2)-i+1/2})$. This is the leading part (again in the sense of [3]) of the Hopf link invariant $\mathcal{W}^{\mathrm{U}(N)}_{R_1R_2}$.

The topological vertex formula derived in [2] reads

$$C_{R_1R_2R_3} = q^{\frac{1}{2}(\kappa_{R_2}+\kappa_{R_3})} \sum_{Q_1,Q_2,R} N^{R_1}_{Q_1R} N^{R_3^t}_{Q_2R} \frac{W_{R_2^tQ_1} W_{R_2Q_2}}{W_{R_2}}\,, \tag{A.5}$$

where κ_R is defined by $\kappa_R = \sum_i l_i(R)(l_i(R)-2i+1)$. Using (A.1) and (A.4) we can express the topological vertex in terms of Schur functions (this was first done in [37])

$$C_{R_1R_2R_3} = q^{\frac{1}{2}(\kappa_{R_2}+\kappa_{R_3})} s_{R_2^t}(q^\rho) \sum_Q s_{R_1/Q}(q^{\ell(R_2^t)+\rho}) s_{R_3^t/Q}(q^{\ell(R_2)+\rho})\,, \tag{A.6}$$

where we have used skew Schur functions defined as

$$s_{R/R_1}(x) = \sum_Q N^R_{R_1Q} s_Q(x). \tag{A.7}$$

Schur functions satisfy some useful identities. First, we have

$$s_{R^t}(q) = q^{-\kappa_R/2} s_R(q) = (-1)^{\ell(R)} s_R(q^{-1})\,, \tag{A.8}$$

and similarly

$$s_{R/R_1}(q) = (-1)^{\ell(R)-\ell(R_1)} s_{R^t/R_1^t}(q)\,. \tag{A.9}$$

The two following formulae are also important:

$$\begin{aligned} \sum_R s_{R/R_1}(x) s_{R/R_2}(y) &= \prod_{i,j\geq 1} (1-x_iy_j)^{-1} \sum_Q s_{R_2/Q}(x) s_{R_1/Q}(y)\,, \\ \sum_R s_{R/R_1}(x) s_{R^t/R_2}(y) &= \prod_{i,j\geq 1} (1+x_iy_j) \sum_Q s_{R_2^t/Q}(x) s_{R_1^t/Q^t}(y)\,. \end{aligned} \tag{A.10}$$

The following result was proved in [10]. Let us define the “relative” hook length

$$h_{R_1R_2}(i,j) = l_i(R_1) + l_j(R_2) - i - j + 1\,, \tag{A.11}$$

and the following functions

$$\begin{aligned} f_R(q) &= \frac{q}{(q-1)}\sum_{i\geq 1}(q^{l_i(R)-i}-q^{-i})\,, \\ \tilde{f}_{R_1R_2}(q) &= \frac{(q-1)^2}{q}f_{R_1}(q)f_{R_2}(q)+f_{R_1}(q)+f_{R_2}(q)\,. \end{aligned} \tag{A.12}$$

Let us denote the expansion coefficients of $\tilde{f}_{R_1R_2}(q)$ by

$$\tilde{f}_{R_1R_2}(q)=\sum_k C_k(R_1,R_2)q^k\,. \tag{A.13}$$

Alternatively,

$$\sum_k C_k(R_1,R_2)q^k=\frac{W_{R_1\square}}{W_{R_1}}\frac{W_{R_2\square}}{W_{R_2}}-W_\square^2\,. \tag{A.14}$$

Then it was proved that

$$\prod_{i,j\geq 1}(1-Qq^{h_{R_1R_2}(i,j)})=\prod_{k=1}^{\infty}(1-Qq^k)^k\prod_k(1-Qq^k)^{C_k(R_1,R_2)}\,. \tag{A.15}$$

Let us now present a useful result proved by Littlewood [29, 30]:

$$\sum_{R=R^T}s_R(x)(-1)^{\frac{1}{2}(\ell(R)\mp r(R))}=\prod_{i=1}^{\infty}(1\pm x_i)\prod_{1\leq i<j<\infty}(1-x_ix_j)\,, \tag{A.16}$$

where $r(R)$ is the rank of R. The final formula that we will need reads as follows [16]

$$\prod_{i,j}(1-Qx_iy_j)=\exp\left[-\sum_{n=1}^{\infty}\frac{Q^n}{n}\sum_{i,j}x_i^ny_j^n\right], \tag{A.17}$$

from which we can deduce the identities

$$\begin{aligned} \prod_i(1\mp Q^{1/2}q^{i-1/2}) &= \exp\left[\sum_{n=1}^{\infty}\frac{(\pm 1)^nQ^{n/2}}{n(q^{n/2}-q^{-n/2})}\right], \\ \prod_{i,j}(1-Qq^{i+j-1}) &= \exp\left[-\sum_{n=1}^{\infty}\frac{Q^n}{n(q^{n/2}-q^{-n/2})^2}\right]. \end{aligned} \tag{A.18}$$

B. A useful identity

In our previous paper [5] we conjectured the relation (3.2) between the topological vertex and SO/Sp Chern-Simons expectation values of Hopf links. This identity showed that the new vertex rule introduced in [5] to compute amplitudes on orientifolds agrees with the results of large N SO/ Sp transitions. In [5] we only presented a partial proof of this relation; we will now present the full proof. We will only consider here the Sp case for the sake of clarity, but the proof for the SO case is similar.

Let us start by considering (3.2) for trivial representations $R_1 = R_2 = \cdot$. In this case we have to show that

$$\sum_{R=R^T} C_{R\cdot\cdot} Q^{\ell(R)/2}(-1)^{\frac{1}{2}(\ell(R)+r(R))} = S_{00}^{\mathrm{Sp}(N)} \,. \tag{B.1}$$

The l.h.s. can be rewritten using (A.6), (A.8) and (A.16) as

$$\begin{aligned}\sum_{R=R^T} s(Q^{1/2}q^{-\rho})(-1)^{\frac{1}{2}(\ell(R)-r(R))} &= \prod_{i=1}^{\infty}(1+Q^{1/2}q^{i-1/2}) \prod_{1\le i<j<\infty}(1-Qq^{i+j-1}) \\ &= \frac{\prod_{i=1}^{\infty}(1+Q^{1/2}q^{i-1/2})^{1/2}\prod_{i,j,=1}^{\infty}(1-Qq^{i+j-1})^{1/2}}{\prod_{i=1}^{\infty}(1-Q^{1/2}q^{i-1/2})^{1/2}}\end{aligned} \tag{B.2}$$

Using (A.18) we find that the r.h.s. becomes

$$\exp\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{((-1)^n-1)Q^{n/2}}{n(q^{n/2}-q^{-n/2})}\right]\exp\left[-\frac{1}{2}\sum_{n=1}^{\infty}\frac{Q^n}{n(q^{n/2}-q^{-n/2})^2}\right] \tag{B.3}$$

which is exactly $S_{00}^{\mathrm{Sp(N)}}$, so (B.1) is proved.

Let now $R_1 = \cdot$ be the trivial representation and $R_2 = \mu$ be any representation. We must now show that

$$\frac{1}{S_{00}^{\mathrm{Sp}(N)}}\sum_{R=R^T} C_{\cdot\mu^T R} Q^{\frac{1}{2}(\ell(R)-\ell(\mu))} q^{\frac{\kappa_\mu}{2}}(-1)^{\frac{1}{2}(\ell(R)+r(R))} = \mathcal{W}_{\mu}^{\mathrm{Sp}(N)} \,. \tag{B.4}$$

Using (A.6) and (A.8) the l.h.s. can be rewritten as

$$\frac{1}{S_{00}^{\mathrm{Sp}(N)}} s_\mu(Q^{-1/2}q^{\rho})\sum_{R=R^T}(-1)^{\frac{1}{2}(\ell(R)-r(R))} s_R(Q^{1/2}q^{-\ell(\mu)-\rho}) \,. \tag{B.5}$$

From (A.16), the first line of (B.2) and the definition of $W_R(q) = s_R(q^\rho)$ in terms of q-numbers (see [2, eq. (7.5)]), we find, after some algebra:

$$\begin{aligned}&Q^{-\ell(\mu)/2}\prod_{1\le i<j\le d(\mu)}\frac{[l_i+l_j-i-j+1]_{Q^{-1}}[l_i-l_j+j-i]}{[-i-j+1]_{Q^{-1}}[j-i]}\times \\ &\times\prod_{i=1}^{d(\mu)}\frac{[1-i]_{Q^{-1}}^{\mathrm{Sp}(N)}[2l_i-2i+1]_{Q^{-1}}}{[l_i+1-i]_{Q^{-1}}^{\mathrm{Sp}(N)}[-2i+1]_{Q^{-1}}}\prod_{v=1}^{l_i}Q^{1/2}\frac{[l_i-i-v-d(\mu)+1]_{Q^{-1}}}{[v-i+d(\mu)]}\,,\end{aligned} \tag{B.6}$$

where $d(\mu)$ is the number of rows of μ, and we used the q-numbers $[x] = q^{x/2} - q^{-x/2}$, $[x]_\lambda = \lambda^{1/2}q^{x/2} - \lambda^{-1/2}q^{-x/2}$ and $[x]_\lambda^{\mathrm{Sp}(N)} = \lambda^{1/4}q^{\frac{1}{4}(2x-1)} - \lambda^{-1/4}q^{-\frac{1}{4}(2x-1)}$. One can see that the two factors of Q cancel out of (B.6), and the remaining expression is exactly the definition of the $\mathrm{Sp}(N)$ quantum dimension of μ for $\lambda = Q^{-1}$ (see [5, eq. (4.9)]). But $\mathcal{W}_\mu^{\mathrm{Sp}(N)} = \dim_q^{\mathrm{Sp}(N)}\mu(\lambda = Q^{-1})$. Therefore (B.4) is proved.

We are now in position to prove (3.2) in the general case, namely we have to show that

$$\frac{1}{S_{00}^{\mathrm{Sp}(N)}}\sum_{R=R^T} C_{R_1 R_2^T R} Q^{\ell(R)/2}(-1)^{\frac{1}{2}(\ell(R)+r(R))} = q^{-\frac{\kappa_{R_2}}{2}} Q^{\frac{1}{2}(\ell(R_1)+\ell(R_2))}\mathcal{W}_{R_1R_2}^{\mathrm{Sp}(N)} \,. \tag{B.7}$$

Let us first rewrite the Hopf link expectation value in terms of quantum dimensions, using [5, eq. (4.19)]. The r.h.s. becomes:

$$\sum_{\mu,\lambda_1,\lambda_2,\lambda_3} N^{R_1}_{\lambda_1\lambda_2} N^{R_2}_{\lambda_1\lambda_3} N^{\mu}_{\lambda_2\lambda_3} q^{\frac{1}{2}(\kappa_{R_1}-\kappa_\mu)} Q^{\frac{1}{2}(\ell(\mu))} \mathcal{W}^{\mathrm{Sp}(N)}_{\mu}, \tag{B.8}$$

where we expressed the Sp tensor product coefficients in terms of Littlewood-Richardson coefficients [12]. We can now rewrite the r.h.s. using (B.4) and (A.6) as

$$\begin{aligned}&\frac{1}{S^{\mathrm{Sp}(N)}_{00}} \sum_{R=R^T} Q^{\frac{1}{2}\ell(R)} (-1)^{\frac{1}{2}(\ell(R)+r(R))} s_{R^T}(q^\rho) q^{\frac{1}{2}(\kappa_{R_1}+\kappa_R)} \times \\ &\times \sum_{\mu,\lambda_1,\lambda_2,\lambda_3} N^{R_1}_{\lambda_1\lambda_2} N^{R_2}_{\lambda_1\lambda_3} N^{\mu}_{\lambda_2\lambda_3} s_{\mu^T}(q^{\ell(R)+\rho}) .\end{aligned} \tag{B.9}$$

The sum in the second line can be explicitely evaluated by using (A.8), (A.9), the definition of skew Schur functions (A.7) and the fact that $s_{R_1}(x)s_{R_2}(x) = \sum_R N^R_{R_1R_2} s_R(x)$:

$$\sum_{\mu,\lambda_1,\lambda_2,\lambda_3} N^{R_1}_{\lambda_1\lambda_2} N^{R_2}_{\lambda_1\lambda_3} N^{\mu}_{\lambda_2\lambda_3} s_{\mu^T}(q^{\ell(R)+\rho}) = \sum_{\lambda_1} s_{R_1^T/\lambda_1^T}(q^{\ell(R)+\rho}) s_{R_2^T/\lambda_1^T}(q^{\ell(R)+\rho}) . \tag{B.10}$$

Inserting (B.10) in (B.9) gives (using the fact that $R = R^T$):

$$\begin{aligned}&\frac{1}{S^{\mathrm{Sp}(N)}_{00}} \sum_{R=R^T} Q^{\frac{1}{2}\ell(R)} (-1)^{\frac{1}{2}(\ell(R)+r(R))} \times \\ &\times \left[q^{\frac{1}{2}(\kappa_{R_1}+\kappa_R)} s_{R^T}(q^\rho) \sum_{\lambda_1} s_{R_2^T/\lambda_1}(q^{\ell(R^T)+\rho}) s_{R_1^T/\lambda_1}(q^{\ell(R)+\rho}) \right] .\end{aligned} \tag{B.11}$$

The term in brackets is exactly the definition of $C_{R_2^T R R_1}$ in terms of Schur functions (see (A.6)). Therefore (B.11) is equal to the l.h.s. of (B.7) and the identity (3.2) is proved.

C. BPS invariants for the trefoil knot

In this appendix, we list the BPS invariants $N^{c=1}_{R,g,\beta}$ for representations R with three boxes.

References

[1] B. Acharya, M. Aganagic, K. Hori and C. Vafa, *Orientifolds, mirror symmetry and superpotentials*, hep-th/0202208.

[2] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *The topological vertex*, *Commun. Math. Phys.* **254** (2005) 425 [hep-th/0305132].

[3] M. Aganagic, M. Mariño and C. Vafa, *All loop topological string amplitudes from Chern-Simons theory*, *Commun. Math. Phys.* **247** (2004) 467 [hep-th/0206164].

[4] M. Aganagic and C. Vafa, *G_2 manifolds, mirror symmetry and geometric engineering*, hep-th/0110171.

[5] V. Bouchard, B. Florea and M. Mariño, *Counting higher genus curves with crosscaps in Calabi-Yau orientifolds*, *J. High Energy Phys.* **12** (2004) 035 [hep-th/0405083].

	$\beta = 2$	3	4	5	6	7	8	9
$g = 0$	18	-270	1185	-2380	2430	-1188	175	30
1	21	-753	4924	-12209	13203	-4856	-1300	970
2	8	-1007	10374	-31348	31419	4028	-22155	8681
3	1	-793	13920	-50383	30636	84956	-117415	39078
4	0	-378	12688	-54222	-24584	305272	-343318	104542
5	0	-106	8006	-40151	-118255	609701	-639896	180701
6	0	-16	3486	-20657	-178503	797521	-813994	212163
7	0	-1	1024	-7353	-161931	728309	-734484	174436
8	0	0	193	-1773	-98947	478948	-480509	102088
9	0	0	21	-276	-42205	229955	-230209	42714
10	0	0	1	-25	-12624	80705	-80729	12672
11	0	0	0	-1	-2599	20474	-20475	2601
12	0	0	0	0	-351	3654	-3654	351
13	0	0	0	0	-28	435	-435	28
14	0	0	0	0	-1	31	-31	1
15	0	0	0	0	0	1	-1	0

Table 4: BPS invariants $N^{c=1}_{\yng(3),g,\beta}$ for the trefoil knot.

	$\beta = 2$	3	4	5	6	7	8	9
$g = 0$	99	-1125	4359	-8096	7828	-3699	563	72
1	201	-4194	22748	-51475	53807	-21649	-2204	2766
2	164	-7702	60811	-165827	171590	-19997	-68978	29939
3	66	-8701	104757	-338906	282625	264688	-468878	164349
4	13	-6395	125047	-472907	124226	1398430	-1710505	542091
5	1	-3092	106648	-466523	-477321	3645201	-3976290	1171376
6	0	-971	65795	-331606	-1232410	6113672	-6363573	1749093
7	0	-190	29358	-171307	-1590490	7192295	-7328205	1868539
8	0	-21	9358	-64261	-1351903	6186865	-6240225	1460187
9	0	-1	2072	-17298	-815116	3979137	-3994110	845316
10	0	0	302	-3252	-358192	1934294	-1937220	364068
11	0	0	26	-405	-115397	712126	-712504	116154
12	0	0	1	-30	-26996	197286	-197315	27054
13	0	0	0	-1	-4465	40454	-40455	4467
14	0	0	0	0	-495	5952	-5952	495
15	0	0	0	0	-33	594	-594	33
16	0	0	0	0	-1	36	-36	1
17	0	0	0	0	0	1	-1	0

Table 5: BPS invariants $N^{c=1}_{\yng(2,1),g,\beta}$ for the trefoil knot.

[6] D.-E. Diaconescu and B. Florea, *Localization and gluing of topological amplitudes*, `hep-th/0309143`.

[7] D.E. Diaconescu, B. Florea and A. Grassi, *Geometric transitions and open string instantons*, *Adv. Theor. Math. Phys.* **6** (2003) 619 [`hep-th/0205234`].

[8] D.E. Diaconescu, B. Florea and A. Grassi, *Geometric transitions, Del Pezzo surfaces and open string instantons*, *Adv. Theor. Math. Phys.* **6** (2003) 643 [`hep-th/0206163`].

[9] D.E. Diaconescu, B. Florea and A. Misra, *Orientifolds, unoriented instantons and localization*, *J. High Energy Phys.* **07** (2003) 041 [`hep-th/0305021`].

	$\beta=2$	3	4	5	6	7	8	9
$g=0$	108	−1044	3705	−6484	6000	−2754	427	42
1	306	−4818	23074	−48785	49436	−20669	−448	1904
2	366	−11012	73663	−186538	193691	−44683	−49616	24129
3	230	−15636	151596	−453623	421630	161750	−421269	155322
4	79	−14720	216949	−756616	429479	1359478	−1836601	601952
5	14	−9381	223615	−898781	−235791	4434624	−5047078	1532778
6	1	−4047	168943	−777340	−1531480	8961515	−9525899	2708307
7	0	−1160	94128	−495542	−2661004	12577678	−12957296	3443196
8	0	−211	38523	−233794	−2843448	12900213	−13087921	3226638
9	0	−22	11409	−81283	−2124814	9936047	−10004126	2262789
10	0	−1	2373	−20525	−1160684	5832726	−5850601	1196712
11	0	0	328	−3656	−470990	2625946	−2629249	477621
12	0	0	27	−435	−142042	905758	−906165	142857
13	0	0	1	−31	−31433	237305	−237335	31493
14	0	0	0	−1	−4959	46375	−46376	4961
15	0	0	0	0	−528	6545	−6545	528
16	0	0	0	0	−34	630	−630	34
17	0	0	0	0	−1	37	−37	1
18	0	0	0	0	0	1	−1	0

Table 6: BPS invariants $N^{c=1}_{\square,g,\beta}$ for the trefoil knot.

[10] T. Eguchi and H. Kanno, *Topological strings and Nekrasov's formulas*, *J. High Energy Phys.* **12** (2003) 006 [hep-th/0310235]; *Geometric transitions, Chern-Simons gauge theory and Veneziano type amplitudes*, *Phys. Lett.* **B 585** (2004) 163 [hep-th/0312234].

[11] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, *Bull. Am. Math. Soc., New Ser.* 12 (1985) 239.

[12] W. Fulton and J. Harris, *Representation theory*, Springer-Verlag.

[13] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

[14] R. Gopakumar and C. Vafa, *M-theory and topological strings, II*, hep-th/9812127.

[15] E. Guadagnini, *The universal link polynomial*, *Int. J. Mod. Phys.* **A 7** (1992) 877.

[16] T.J. Hollowood, A. Iqbal and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, hep-th/0310272.

[17] A. Iqbal and A.K. Kashani-Poor, *Instanton counting and Chern-Simons theory*, *Adv. Theor. Math. Phys.* **7** (2004) 457 [hep-th/0212279].

[18] A. Iqbal and A.-K. Kashani-Poor, *The vertex on a strip*, hep-th/0410174.

[19] S. Katz and C.C.M. Liu, *Enumerative geometry of stable maps with lagrangian boundary conditions and multiple covers of the disc*, *Adv. Theor. Math. Phys.* **5** (2002) 1 [math.ag/0103074].

[20] L.H. Kauffman, *An invariant of regular isotopy*, *Trans. Am. Math. Soc.* 318 (1990) 417

[21] J.M.F. Labastida, P.M. Llatas and A.V. Ramallo, *Knot operators in Chern-Simons gauge theory*, *Nucl. Phys.* **B 348** (1991) 651.

[22] J.M.F. Labastida and M. Marino, *The homfly polynomial for torus links from Chern-Simons gauge theory*, *Int. J. Mod. Phys.* **A 10** (1995) 1045 [hep-th/9402093].

[23] J.M.F. Labastida and M. Marino, *Polynomial invariants for torus knots and topological strings*, *Commun. Math. Phys.* **217** (2001) 423 [hep-th/0004196].

[24] J. M. F. Labastida and M. Mariño, *A new point of view in the theory of knot and link invariants*, *J. Knot Theory Ramifications* **11** (2002) 173 [math.qa/0104180]

[25] J.M.F. Labastida, M. Mariño and C. Vafa, *Knots, links and branes at large-N*, *J. High Energy Phys.* **11** (2000) 007 [hep-th/0010102].

[26] J.M. F. Labastida and E. Pérez, *A relation between the kauffman and the HOMFLY polynomials for torus knots*, *J. Math. Phys.* **37** (1996) 2013 [arXiv:q-alg/9507031]

[27] J. Li, C.C. Liu, K. Liu and J. Zhou, *A mathematical theory of the topological vertex*, math.ag/0408426.

[28] W.B.R. Lickorish, *An introduction to knot theory*, Springer-Verlag, 1998.

[29] D.E. Littlewood, *The theory of group characters and matrix representations of groups*, Oxford, The Clarendon Press, 1940, p. 292.

[30] I.G. Macdonald, *Symmetric functions and hall polynomials*, Oxford Mathematical Monographs, Oxford Science Publications, 1995, second edition.

[31] M. Mariño, *Chern-Simons theory and topological strings*, hep-th/0406005.

[32] M. Mariño and C. Vafa, *Framed knots at large-N*, hep-th/0108064.

[33] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, I*, math.ag/0312059; *Gromov-Witten theory and Donaldson-Thomas theory, II*, math.ag/0406092.

[34] H.R. Morton and S.G. Lukac, *The HOMFLY polynomial of the decorated Hopf link*, *J. Knot Theory Ramifications* **12** (2003) 395 [math.GT/0108011].

[35] S.G. Naculich, H.A. Riggs and H.J. Schnitzer, *Simple current symmetries, rank level duality and linear skein relations for Chern-Simons graphs*, *Nucl. Phys.* **B 394** (1993) 445 [hep-th/9205082].

[36] A. Neitzke and C. Vafa, *Topological strings and their physical applications*, hep-th/0410178.

[37] A. Okounkov, N. Reshetikhin and C. Vafa, *Quantum Calabi-Yau and classical crystals*, hep-th/0309208.

[38] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys.* **B 577** (2000) 419 [hep-th/9912123].

[39] P. Rama Devi, T.R. Govindarajan and R.K. Kaul, *Three-dimensional Chern-Simons theory as a theory of knots and links, 3. Compact semisimple group*, *Nucl. Phys.* **B 402** (1993) 548 [hep-th/9212110].

[40] S. Sinha and C. Vafa, SO *and* Sp *Chern-Simons at large-N*, hep-th/0012136.

[41] C. H. Taubes, *Lagrangians for the Gopakumar-Vafa conjecture*, *Adv. Theor. Math. Phys.* **5** (2001) 139 [math.DG/0201219]

[42] E. Witten, *Quantum field theory and the jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351.